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XXXIX. *Theory of the Internal Action of Thermionic Systems at Moderately High Frequencies.*—Part II. *By* W. E. BENHAM, *B.Sc.**

ERRATA, CORRIGENDA, AND ADDENDA TO PART I.
(Phil. Mag. v. p. 641, March 1928).

P. 643. The factor $\frac{1}{4\pi}$ should multiply the last term of equation (2 *b*).

P. 644. Introducing equation (6), *read* "subject to $U=0$ and $U\frac{dU}{dx}=0$".

P. 649. Equation (23). "161" should *read* "168."

P. 659. 17 lines from bottom of page *read* "theory for *zero* initial velocities" *instead of* "theory for initial velocities."

The above errors are typographical and of no consequence. The only error of importance occurs in equation (14). I am indebted to Mr. B. D. H. Tellegen of Eindhoven for pointing out the error in a communication dated 25th April, 1929, of which the following is an extract :—

"In working through your calculations I came to a different result as regards the influence of the frequency on

* Communicated by the Author.

the rectification. In your paper the last term of the left-hand side of equation (14) is stated as

$$2\frac{\partial}{\partial t}\left(u\frac{\partial u}{\partial x}\right).$$

"I come to

$$\frac{\partial}{\partial t}\left(u\frac{\partial u}{\partial x}\right)+u\frac{\partial^2 u}{\partial x\partial t}.$$

"Starting from this form of equation (14), the expression between round brackets of equation (25) becomes

$$1-\frac{3\xi^2}{80}+\frac{\xi^4}{1400}.$$

"In equation (30) the expression between round brackets becomes then

$$1-\frac{\xi^2}{40}+\frac{\xi^4}{2800},$$

and the final equation for $\frac{\hat{i}_0'}{2}$ of p. 651 becomes

$$\frac{\hat{i}_0'}{2}=\left(\frac{3I_0}{16}\right)\frac{v_d^2}{V_d^2}\left(1+\frac{11\xi_d^2}{600}+\frac{39\xi_d^4}{140{,}000}\right).$$

The influence of the frequency on the rectification has now reversed its sign."

The theoretical curve of fig. 1 (p. 652) is thus erroneous. In view of the remarks at the bottom of p. 655, however, the consequences are, fortunately, not serious. The correct curve will be found on fig. 17 of this paper.

The indications of a further analysis are

1. That in the case of cylindrical electrodes the rectification *does* decrease with frequency.
2. That in the case of voltage saturation a decrease with frequency takes place which is the same for planes and cylinders alike.

In view of the fact that cylindrical structures were employed in the experiments the solution required is more nearly that which emerges from the consideration of cylinders given at the end of this paper. It must not be thought, however, that the mere difference in sign of a very small coefficient in the "second order" solution signifies any fundamental difference between the behaviour of the space-charge in the straightforward plane case, on the one hand, and the more complex cylindrical case on the other. The

"first order" solutions in the two cases are, in fact, so similar that valuable conclusions of a general character may be derived from a detailed study of the plane case.

Contents.

1. *Introduction.*

THE theory outlined in Part I. contains just sufficient analysis for the purposes of the "second order" solution and the extraction therefrom of an expression for the rectification effect as a function of pT. The sequel aims at an exhaustive analysis of the "first order" solution. The theory is still subject to the assumptions made in Part I., which will be repeated here :

(1) The propagation of electromagnetic waves across the space takes place in a time which is small compared with the time period of the potential and time of transit T of the electrons.

(2) The alternating potential is small compared with the steady potential existing between the plates.

(3) The steady state distribution is that for which initial velocities are neglected.

These are the only assumptions we shall need. As to the effect of these assumptions, the least serious is (1), and confines the theory to frequencies below about 10^{10} cycles per second in the case of plates 1 mm. apart. This limitation may be regarded as defining the term "moderately high frequency" as used in the title of the paper. With regard to (2), the limitation introduced is again not very serious. All the important valve "constants" imply very small changes, otherwise they would have no meaning owing to the curvature of the characteristic. The only assumption likely to influence the quantitative accuracy of the theory is (3), and even here the main burden of error is thrown on the "d.c." part of the solution, the important quantities for our purposes being functions only of pT *, having in all probability a high order of accuracy within the limitations of assumption (2).

2. *Scope of Part II.*

The theory is used to replace the thermionic system (as far as its first order a.c. properties are concerned) by an equivalent circuit, which may be defined as follows :—

An electrical network which, when used to replace the thermionic system, gives results which are indistinguishable from those given by the thermionic system by measurements in any associated circuit.

In general the equivalent circuit contains four components, but the circuit reduces to one having only two components in the particular case of two electrode systems. From the values of the "equivalent circuit" components the admittance conductance and susceptance of a parallel plane diode are evaluated as functions of pT. The conductance is negative at certain values of pT.

Considerable space is devoted to the subject of the dielectric constant of space-charge, and experiments are quoted which support the theoretical conclusions. The space-charge case

* T is the time of transit in the absence of the alternating component, and may be regarded as the mean value of the time of transit in the presence of *small* alternations.

is compared and contrasted with the case of a layer of electrons in loose association with positive ions.

The impedance of the equivalent circuit is always capacitative. The negative value of conductance at certain values of pT is discussed in its bearing on the hypothesis that the negative resistance property is inherent in the space-charge itself. This idea has been held, but to the best of my knowledge never proved, by several authors (*cf.* Gill and Morell, Phil. Mag. (6) xliv. p. 162 (1922) ; Karl Kohl, *Ann. d. Phys.* ccccIxxxv. p. 1 (1928) ; Lewi Tonks, Phys. Rev. (2) xxix. p. 913 (1927)).

The conditions for the generation of oscillations in a parallel plane are such that oscillations would only be obtained from a diode in exceptional circumstances. Artificial lengthening of the time of transit (perforated anode, gas, magnetic field) is in all cases favourable to the production of oscillations by a diode. A mathematical transition to the complete triode case is not attempted, but the diode theory nevertheless contains an explanation of B.-K. oscillations*. It also provides a lower limit for the time period of B.-K. oscillations, which explains the observations of B. and K. using a high filament temperature.

A simple treatment neglecting space-charge confirms that there is no negative conductance in this case, showing that the negative conductance is a property resulting from the mutual repulsions of electrons. The frequency variation of the rectification effect with and without space-charge is discussed.

An outline of the cylindrical case is given, and a solution derived in the particular case $\frac{R}{a} \to \infty$. The solution for this case is of the same order of magnitude as that for the plane case, and shows a marked similarity of graphical, though not of analytical form. An approximate solution is obtained for finite values of $\frac{R}{a}$, which yields an expression for the dielectric constant of space-charge between cylindrical electrodes.

3. In the equations $(1a)-(4a)$, p. 645 of Part I., let us work with the *total* current J instead of the current I (hereafter referred to as the *conduction* current).

* Barkhausen and Kurz, *Phys. Zeit.* xxi. p. 1 (1920).

We have then *

$$\frac{\partial^2 \mathbf{V}}{\partial x^2} = -4\pi \mathbf{P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\mathbf{J} = \mathbf{PU} - \frac{1}{4\pi} \frac{\partial^2 \mathbf{V}}{\partial x \partial t}, \quad . \quad . \quad . \quad (2)$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \frac{\partial \mathbf{U}}{\partial x} = -\frac{e}{m} \frac{\partial \mathbf{V}}{\partial x}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{\partial \mathbf{J}}{\partial x} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which $\mathbf{V}$, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{U}$ are instantaneous values of potential, space-charge per unit volume, current, density, and velocity respectively.

The elimination of $\mathbf{V}$ and $\mathbf{P}$ leads directly to the equation

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \frac{\partial}{\partial x}\right)\left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \frac{\partial \mathbf{U}}{\partial x}\right) = 4\pi \frac{e}{m} \mathbf{J}. \quad . \quad . \quad (5)$$

The treatment given in Part I. led to the slightly different equation

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \frac{\partial}{\partial x}\right)\left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \frac{\partial \mathbf{U}}{\partial x}\right) = 4\pi \frac{e}{m} \mathbf{I}_0. \quad . \quad . \quad (5\,a)$$

Equation (5) is independent of any hypothesis as to the value of the field at the cathode. It is therefore a more general starting point than equation (5 *a*) which is subject to the condition $\left|\frac{dV}{dx}\right|_0 = 0$. This condition, however, appears later owing to the use of $U = ax^{2/3}$, p. 647, Part I., so that the gain in generality is slight only.

If we now reintroduce the condition $\left|\frac{dV}{dx}\right|_0 = 0$, equations (5) and (5 *a*) can be directly compared to give us

$$\mathbf{I}_0 = \mathbf{J}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

meaning that the displacement current $\mathbf{Y}(=\mathbf{J}-\mathbf{I})$ has zero value at the cathode if we take $\left|\frac{dV}{dx}\right|_0 = 0$, and not, as Muller and Tank† suppose, that there is "no displacement current of the electrodes." This important distinction will

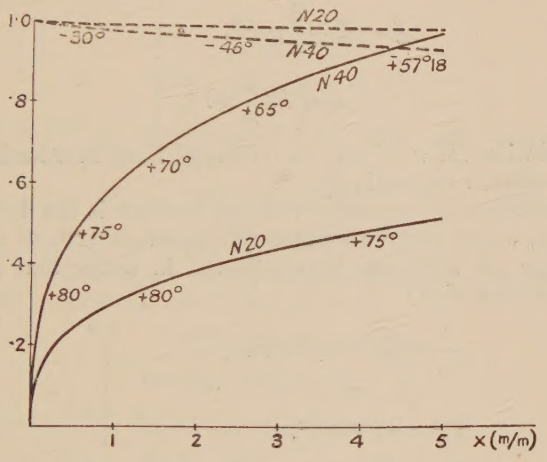
* The non-appearance of the magnetic field in the space-charge equations is accounted for by the fact that the second order magnetic and electric forces cannot affect variations in the direction of motion.

† Muller and Tank, *Helvetica Physica Acta*, i. p. 447 (1928).

be better understood from fig. 1, on which the moduli of displacement and conduction currents appear as functions of x. Fig. 1 reveals that the displacement current is finite in general, though zero at the cathode. When due allowance is made for the varying phase angle (indicated on the curve), it will be found that fig. 1 is in keeping with the condition that the sum of the *instantaneous* values of displacement and conduction current must be independent of x.

Equation (5), being a partial differential equation of the second order and of the third degree in **U**, does not lend

Fig. 1.



Showing the variation with x of the moduli of conduction current (broken curves) and of displacement current (full-line curves) between electrodes 5 mm. apart subject to 16 volts d.c. potential difference, superimposed on which is a small oscillatory p.d. of frequency **N** megacycles. Frequencies N20 and N40 are represented. Modulus of total current is everywhere unity. Phase angles marked at intervals are with respect to total current. Where the curves intersect the phases of the two currents are equal and opposite.

itself to direct solution. By subdividing the independent variables according to the scheme

$$\left.\begin{aligned} \mathbf{U} &= \mathrm{U} + u + u' + \ldots \\ \mathbf{J} &= \mathrm{J} + j + j' + \ldots \end{aligned}\right\}^{*},$$

where the principal quantity U is a function only of x, and

* Physically **J** is of the nature of a disturbance, **U** being the effect. The advantage of treating **J** rather than **V** as the disturbance is largely a question of analysis.

u, u' ... are of the first, second, ... order of small quantities, the equation may be made to yield a linear equation in u and another linear equation in u'.

(N.B.—Fluctuating components are denoted by small letters, while the symbols V, P, J, U, not in Clarendon type, are retained for principal quantities.)

The linear equation in u may be solved in terms of j, x, and t, provided j is a known function of t and U is a known function of x. The linear equation in u' may be solved provided j' is a known function of t and u is a known function of x and t. The solution is preferably obtained in terms of ξ and t, where

$$\left.\begin{aligned} \xi = p\mathrm{T} = \frac{3}{\alpha} p x^{1/3}, \\ \alpha = \left(18 \frac{e}{m} \pi \mathrm{J}\right)^{1/3}. \end{aligned}\right\}$$

Any solution for u, u' will be referred to as first and second order solution respectively.

The solution of greatest practical interest is the first order solution, which may be written (*cf.* equation (20) of Part I.; a change of sign has been effected in order that Ξ may carry its own sign)

$$\left.\begin{aligned} &\frac{u}{\hat{u}_0} = \Upsilon_u \sin(pt + \Xi_u), \\ &\text{where} \\ &\Upsilon_u^2 = \mathrm{S}_u^2 + \mathrm{S}'^2_u, \\ &\mathrm{S}_u = \frac{6}{\xi^2}\left(-\cos\xi + \frac{2}{\xi}\sin\xi - 1\right), \\ &\mathrm{S}_u' = \frac{6}{\xi^2}\left(\sin\xi + \frac{2}{\xi}\cos\xi - \frac{2}{\xi}\right), \\ &\Xi_u = -\frac{1}{2}\xi. \end{aligned}\right\}^{*} \quad \ldots \quad (6)$$

The above solution may be used to evaluate the conduction current I, using equation (7) of Part I., *e. g.*,

$$\mathrm{U}\frac{\partial}{\partial x}\left(\frac{\partial \mathrm{U}}{\partial t} + \mathrm{U}\frac{\partial \mathrm{U}}{\partial x}\right) = 4\pi \frac{e}{m}\mathrm{I} \quad \ldots \quad (7)$$

In what follows the word "current" implies "the fundamental component of *fluctuating* current" (small letters).

* The suffix "0" here refers to $\xi = 0$.

The statement

total current = conduction current + displacement current

may be written in mathematical form as follows :

$$\hat{j}\sin pt = \hat{i}\sin(pt+\Xi_i)+\hat{y}\sin(pt+\Xi_y). \quad . \quad . \quad . \quad (8)$$

Now $\hat{i}$ and Ξ_i are completely known, in view of (6) and (7), and determine values of the conduction current relative to the total current $\hat{j}\sin pt$.

Rewriting (8) in the form

$$\sin pt = \Upsilon_i \sin(pt+\Xi_i)+\Upsilon_y \sin(pt+\Xi_y),$$

where

$$\left.\begin{aligned}\Upsilon_i &= \hat{i}/\hat{j},\\ \Upsilon_y &= \hat{y}/\hat{j},\end{aligned}\right\}$$

we now define quantities S_r, S_r' such that, whatever the significance of the suffix r,

$$\left.\begin{aligned}S_r^2+S_r'^2 &= \Upsilon_r^2,\\ S_r'/S_r &= \tan\Xi_r.\end{aligned}\right\}$$

The relation of the displacement current to the conduction current is then seen to be as follows :

$$\left.\begin{aligned}\Upsilon_y^2 &= 1+\Upsilon_i^2-2S_i,\\ \tan\Xi_y &= S_i'/(1-S_i).\end{aligned}\right\}$$

Values of Υ_i^2, S_i, S_i' are as follows :

$$\left.\begin{aligned}\Upsilon_i^2 &= \frac{4}{\xi^4}[\xi^2-2\xi\sin\xi+2(1-\cos\xi)],\\ S_i &= \frac{2}{\xi^2}[\xi\sin\xi-1+\cos\xi],\\ S_i' &= \frac{\xi}{\xi^2}[\xi\cos\xi-\sin\xi],\\ \text{where}\quad \xi &= p\mathrm{T}.\end{aligned}\right\} \quad . \quad . \quad (10)$$

Equations (10) and (9) determine the conduction and displacement current at any frequency $p/2\pi$ and at any distance $x\left(=\frac{2e}{3m}\pi\mathrm{I}\mathrm{T}^3\right)$ from the cathode. (It is noteworthy that Ballantyne * has derived by a different method, in which space-charge was not taken into account, a formula for the

* Ballantyne, *J. Frankl. Inst.* ccvi. no. 2, p. 159, Aug. 1928.

frequency variation of the Schottky effect which is identical with the above expression for Υ_i^2. (*Cf.* section 18.)

5. Corresponding expressions are readily obtainable for the fluctuating components of space-charge and potential. The complete expressions are given in Appendix I. For present purposes simplified expressions suffice, *e. g.*, the first three terms of the series expansions in ascending powers of ξ These differ only slightly from the exact expressions for small values of ξ, the difference amounting to 1 per cent. only in the case of Υ_i at $\xi=\pi$. Only for mathematical reasons is Υ_r^2 given rather than Υ_r in Table I. The second

TABLE I.

Physical Quantity.	Symbol.	Υ_r^2.	$\frac{\Upsilon_r^2}{\Upsilon_v}$.
Current (total)	—	1	$1+\frac{13}{300}\xi^2$
Space-charge ...	Υ_ρ^2	$\left(1-\frac{3}{80}\xi^2+\frac{1}{1400}\xi^4\right)$	$1+\frac{7}{1200}\xi^2$
Potential.....	Υ_v	$\left(1-\frac{13}{300}\xi^2+\frac{11}{12,600}\xi^4\right)$	1
Velocity	Υ_u	$\left(1-\frac{1}{20}\xi^2+\frac{3}{2800}\xi^4\right)$	$1-\frac{1}{150}\xi^2$
Conduction current.	Υ_i	$\left(1-\frac{1}{18}\xi^2+\frac{1}{720}\xi^4\right)$	$1-\frac{11}{900}\xi^2$
Displacement current.	Υ_y	$\frac{4}{9}\xi^2\left(1-\frac{19}{320}\xi^2\right)$	$\frac{4}{9}\xi^2\left(1-\frac{77}{4800}\xi^2\right)$

column gives two terms only of the expansion after dividing by Υ_v^2, which process is necessary in order to convert to a constant oscillatory *voltage* input.

Table II. gives the arguments Ξ_r and $\Xi_r-\Xi_v$ in radians. The second and fourth columns give arguments in the particular case $\xi=1{\cdot}507$, for which value of ξ, $\Upsilon_y=\Upsilon_i=0{\cdot}938$ (see fig. 1). The fifth column converts the figures of the ourth column to degrees and minutes.

That the total current "leads" the potential in phase at all frequencies is due entirely to the displacement current, the "lag" of the conduction current being always too small to counteract the capacitative "lead." It will be noticed

that $\Xi_\rho < \Xi_i < \Xi_u$, as would be expected from equation I = PU. The difference between Ξ_ρ and Ξ_v arises from the dual nature of the potential *, and represents the small interval of time which must elapse between the application of the field and the establishment of the space-charge potential. The value of $-\Xi_v$ never greatly exceeds $\frac{\pi}{2}$, as may be seen by the polar diagram of fig. 2. It follows that, as far as the external circuit is concerned, a thermionic system behaves

TABLE II.

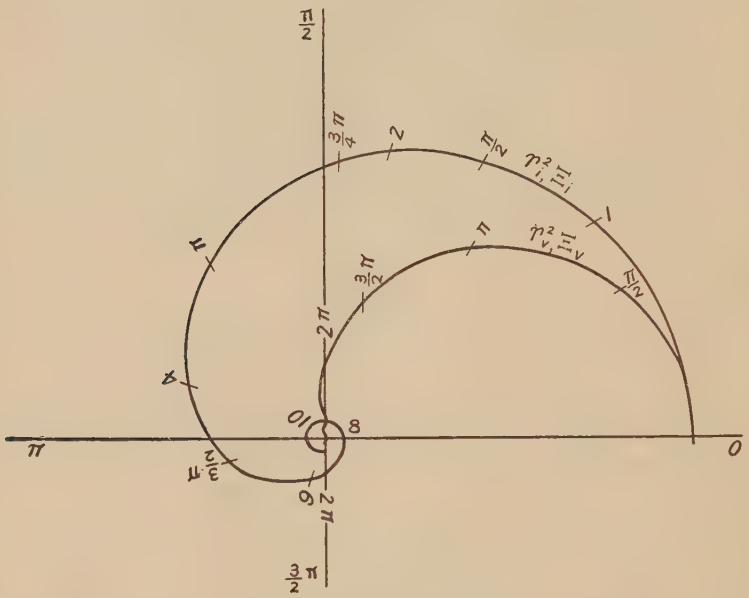
		1.	2.	3.	4.	5.
Physical Quantity.	Symbol.	Ξ_r.	Ξ_r at $\xi=1{\cdot}507$.	$\Xi_r - \Xi_v$.	$\Xi_r - \Xi_v$ at $\xi=1{\cdot}507$.	
Displacement current.	Ξ_y	$\frac{\pi}{2} - \frac{3}{8}\xi$	+1·005	$\frac{\pi}{2} - \frac{3}{40}\xi$	+1·457	+83° 29′
Total current.	—	0	0	$+\frac{3}{10}\xi$	+ ·452	+25° 54′
Potential.....	Ξ_v	$-\frac{3}{10}\xi$	−·452	0	0	0
Velocity	Ξ_u	$-\frac{1}{2}\xi$	−·753	$-\frac{1}{5}\xi$	− ·311	−17° 48′
Conduction current.	Ξ_i	$-\frac{2}{3}\xi$	−1·005	$-\frac{11}{30}\xi$	− ·553	−31° 41
Space-charge.	Ξ_ρ	$-\frac{3}{4}\xi$	−1·130	$-\frac{9}{20}\xi$	− ·678	−38° 51

always as a *capacity* and never as an *inductance* (paralleled by a resistance). This state of affairs is represented on fig. 3*a*. It will appear later that the equivalent circuit of fig. 3*a* is only applicable to a diode, *i. e.*, is not true in general (see section 11).

From Table I. we note that the admittance of the system increases with the frequency, the admittance being proportional to $\frac{1}{\Upsilon_v}$. This increase with frequency only arises, however, because of the presence of displacement current.

* Kimball, Phil. Mag. xlix. p. 695 (1925).

Fig. 2.



Explanatory Note.—The angle between the radius vector and the initial line is Ξ_v in one case and Ξ_i in the other. This method of plotting is slightly preferable to th t using ξ as argument, as it constitutes a graphical representation of the relation between modulus and argument of the voltage and conduction current respectively. From the values of ξ marked at intervals on both curves the relation between Ξ and ξ may be seen in the two cases.

Fig. 3*a*.

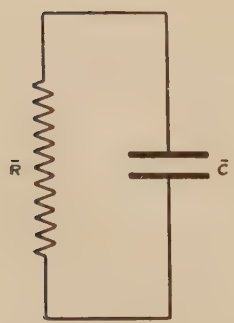
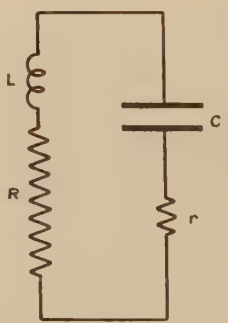


Fig. 3*b*.



If we divide the admittance into its component parts, conductance and susceptance, the latter can be represented by the equation

$$p\overline{\mathrm{C}} = \frac{1}{\overline{\mathrm{R}}}\tan^{-1}(-\Xi_v)$$

$$= \frac{-\mathrm{S}_v'}{\overline{\mathrm{R}}\mathrm{S}_v},$$

where $\overline{\mathrm{C}}$, $\overline{\mathrm{R}}$ are effective values of capacity resistance in the equivalent circuit of fig. 3 *a*.

Now the "conduction current" is not in phase with the potential, hence we cannot write Υ_i/Υ_v for the conductance. We proceed as follows.

6. The instantaneous rate of expenditure of energy by the *alternating* components of current and potential is

$$(\bar{\mathrm{J}} + \hat{j}\sin pt)\,[\mathrm{V} + \hat{v}_0(\mathrm{S}_v \sin pt + \mathrm{S}_v' \cos pt)\,] - \mathrm{JV}.$$

Omitting terms which integrate to zero over a time period, the increase in "power" is

$$\hat{j}\hat{v}_0 \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} \mathrm{S}_v \sin^2 pt\, dt = \tfrac{1}{2}\hat{j}\hat{v}_0\mathrm{S}_v. \quad . \quad . \quad . \quad (11)\,*$$

This is apart from that arising from the rectified current (see section 7). The energy expenditure at zero frequency is obtained by putting $\mathrm{S}_v = 1$. The above is subject to $\hat{j}$ constant. For $\hat{v}$ constant we replace $\hat{j}$ and S_v by $\hat{j}_0/\Upsilon_v$ and S_v/Υ_v respectively.

Let R_0 be the value of $\overline{\mathrm{R}}$ at zero frequency. Then $\hat{j}_0\mathrm{R}_0 = v_0$, since at zero frequency the parallel capacity takes no current. We then have for the h.f. energy expenditure

$$\frac{1}{2}\frac{\hat{v}^2}{\overline{\mathrm{R}}} = \frac{1}{2}\frac{\hat{v}_0^2}{\mathrm{R}_0} \cdot \frac{\mathrm{S}_v}{\Upsilon_v^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Since now $\hat{v} = \hat{v}_0$, we have

$$\frac{\overline{\mathrm{R}}}{\mathrm{R}_0} = \frac{\Upsilon_v^2}{\mathrm{S}_v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

* In arriving at this formula due allowance has been made for the fact that $\hat{j}$ includes the displacement current. We may correctly write $\frac{1}{2}\hat{j}\hat{v}_0\mathrm{S}v = \frac{1}{2}\hat{j}^2\mathrm{R}_0\mathrm{S}_v$, but now since $\hat{j}$ appears twice over it would be incorrect to interpret $\mathrm{R}_0\mathrm{S}_v$ as the resistance. We may, however, define $\mathrm{R}_0\mathrm{S}_v$ as *the effective resistance to the current j*. *Radiation resistance* is similarly defined. See Appendix IV.

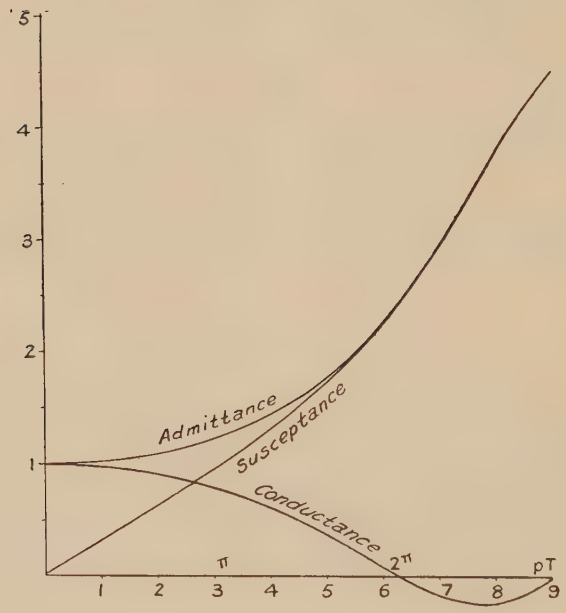
7. The susceptance then becomes

$$p\bar{C} = \frac{-S_v'}{R_0 \Upsilon_v^2}, \quad \ldots \ldots \quad (14)$$

while the conductance is

$$\frac{1}{\bar{R}} = \frac{S_v}{R_0 \Upsilon_v^2}. \quad \ldots \ldots \quad (15)$$

Fig. 4.



As a check on these results, we note that the admittance is given by

$$\left(p^2\bar{C}^2 + \frac{1}{\bar{R}^2}\right)^{1/2},$$

which leads to the correct result, namely

$$\frac{1}{R_0 \Upsilon_v}. \quad \ldots \ldots \quad (16)$$

The admittance, conductance, and susceptance are plotted as fractions of $\frac{1}{R_0}$ on fig. 4 for values of ξ up to 9.

The negative value of the h.f. energy expenditure arising from the negative conductance manifested between $\xi 2\pi$ and $\xi 9$ indicates the possibility of oscillations in diodes (*cf.* section 17). The energy required for such oscillations must be derived from the battery. It is easily shown that the increase in anode current as a result of rectification absorbs energy from the battery of amount

$$\frac{1}{2}\frac{\dot{v}_0^2}{R_0}\left(\frac{T_i'}{4T_v^2}\right),$$

but this energy is, of course, dissipated as heat. When S_v is negative

$$\left(-\frac{1}{2}\frac{\dot{v}_0^2}{R_0}S_v\right)$$

is radiated away as h.f. energy *.

We thus see that *the transformation of d.c. into a.c. energy is characteristic of the simplest possible example of thermionic system, and this without the assistance of an external circuit.* An external circuit of some sort is, however, always present, and must be allowed for in considering the possibility of oscillations in practice (*cf.* section 17).

8. Knowing the susceptance $p\bar{C}$, let us seek the value of $\bar{C}$ in terms of the capacity C_1 of the system when cold. C_1, which is reckoned per unit area of plate, will hereafter be referred to as the "cold capacity".

We have

$$I = KV^{3/2}/d^2,$$

where d is the value of x at the anode ;

$$\frac{1}{R_0} = \frac{dI}{dV} = \frac{3}{2}KV^{1/2}/d^2$$
$$= 1/2\pi d.T.$$

But

$$C_1 = 1/4\pi d ;$$

hence we obtain

$$\frac{T}{R_0} = 2C_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The above result may be stated as follows:—"Provided there is ample emission, the ratio of the time of transit to the static impedance is a constant, and equal to twice the cold capacity." The expression

$$T = 2\times 10^{-12}C_1R_0, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

* Provided certain conditions be fulfilled. See Appendix IV.

in which C_1 is in micromicrofarads and R_0 in ohms, may be found useful for calculations of T. In this connexion it may be noted that plates 0·0885 cm. apart have a capacity of 1 micromicrofarad per unit area; (18) holds for plates of any area if C_1 be the capacity of the area, C_1R_0 being independent of the area of plate.

Returning to the calculation of $\overline{C}$, we have from equations (14) and (17)

$$p\overline{C} = \frac{2C_1}{T}\left(\frac{-S_v'}{\Upsilon_v^2}\right)$$

or

$$\frac{\overline{C}}{C_1} = \frac{2}{\xi}\left(\frac{-S_v'}{\Upsilon_v^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

At zero frequency this reduces to

$$\frac{\overline{C}_0}{C_1} = \frac{3}{5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The value of the "dielectric constant of space-charge" thus comes out to be less than unity. It is noteworthy that the dielectric constant cannot be calculated by evaluating the charge on the anode and dividing by the potential *. The result obtained by this method, namely $\frac{4}{3}$, has, however, certain significance, as will afterwards appear. The difference between Heaviside layer and an atmosphere of space-charge will be discussed later (see section 16). The ratio $\frac{\overline{C}}{C_1}$ is plotted on fig. 5 *a* for values of ξ up to 6π. It is seen that maxima occur at odd multiples of π, except π itself, the maximum possible value of $\overline{C}$ being C_1. The minima occur at even multiples of π, except 2π. The value of $\frac{\overline{C}}{C_1}$ approaches unity in a series of oscillations of small and diminishing amplitude. At technical frequencies the value

* Owing to implicit neglect of the inertia. The electron inertia is, however, taken into account if an energy basis is taken for calculation. Thus

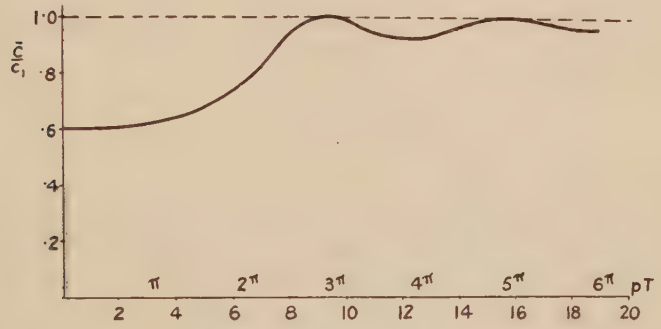
$$\frac{\epsilon f^2}{8\pi} = \frac{1}{d}\int_0^d \frac{f^2}{8\pi}dx,$$

in which the field $f \propto x$. Integration yields $\epsilon = \frac{3}{5}$ for the effective dielectric constant of the volume *d*.

of $\frac{\bar{C}}{C_1}$ would not, as a rule, exceed 0·6 by more than a few per cent.

The result $\frac{\bar{C}}{C_1} = \frac{3}{5}$ only applies under the space-charge conditions stipulated, and, as will be seen in section 12, in cases where one of the electrodes is the negative grid of a triode we should expect $\frac{\bar{C}}{C_1} > 1$ (whatever the form of the electrodes and under all space-charge conditions). Experimental evidence that $\frac{\bar{C}}{C_1} > 1$ under these conditions is not lacking *. Ballantyne † states that the geometric capacity is

Fig. 5 *a*.



augmented by an extra *capacity* effect due to the *gross motion* of the statistical space-charge. The italics are mine, as it is just the "gross motion" of the space-charge which (except in the case above cited) more than counteracts any increase in effective capacity due to the mere presence of the space-charge.

I should like to amend Ballantyne's statement, to read as follows:—"The geometric capacity is augmented by a capacity effect due to the presence of the space-charge, but the gross motion of the space-charge gives rise to an inductive effect which may diminish the effective capacity."

* *Cf.* Hartshorn, Proc. Phys. Soc. xli. 2, no. 227, p. 118 (Feb. 1929). In this work $C_{ga}+C_{fa}$ was measured. Since $C_{fa} << C_{ga}$, the capacity measured was mainly that between grid and anode.

† *Loc. cit.*

The question as to the magnitude of the inductive effect is easily settled. Thus for ξ small

$$\Xi_v - \Xi_i = -\frac{3}{10}\xi + \frac{2}{3}\xi = +\frac{11}{30}\xi.$$

Let us represent the effect by introducing a small inductance L in series with the resistance R of the system (*cf.* section 9). At low frequencies (suffix 0), we have

$$\frac{pL_0}{R_0} = \frac{11}{30}\xi,$$

or, placing $\xi = pT$ and using (17),

$$L_0 = \frac{11}{15} C_1 R_0^2. \quad \ldots\ldots \quad (21)$$

The apparent reduction of capacity arising from the presence of this inductance is $\frac{11}{15} C_1$ at zero frequency. The *true* capacity C_0 at zero frequency is thus

$$\left(\frac{3}{5} + \frac{11}{15}\right) C_1 = \frac{4}{3} C_1.$$

9. For the purpose of arriving at a true understanding of the nature of a thermionic system it is of advantage to keep the true capacity separate from the inductance. We are thus led to fig. 3 *b*, in which L, R, and C have obvious significance. Fig. 3 *b* may be regarded as the true "equivalent circuit." The relative merits of figs. 3 *a* and 3 *b* are discussed in section 11. The resistance r is necessary, in view of the fact that no real value of C satisfies the relation

$$y = -C\frac{dv}{dt}.$$

While, strictly speaking, the losses in the electron capacity are distributed, the effect is that of a resistance in series with a pure capacity *. Similar remarks apply to the inductance L in series with the pure resistance R. We now proceed to calculate these quantities.

* It would, alternatively, be possible to place a conducting path in parallel with C, but then we should have three parallel branches to consider. In this case, therefore, a series resistance is to be preferred, though in circuit analysis the shunted conductance method is more usual (Steinmetz, 'Transient Electric Phenomena and Oscillations').

(*a*) L, R branch :

$$\left.\begin{aligned} \mathrm{R}i+\mathrm{L}\frac{di}{dt} &= v, \\ (\mathrm{R}\mathrm{S}_i-p\mathrm{L}\mathrm{S}_i') &= \mathrm{R}_0\mathrm{S}_v, \\ (\mathrm{R}\mathrm{S}_i'+p\mathrm{L}\mathrm{S}_i) &= \mathrm{R}_0\mathrm{S}_v', \\ \frac{\mathrm{R}}{\mathrm{R}_0} &= \frac{\chi_{\mathrm{R}}}{\Upsilon_i^2}, \\ \frac{p\mathrm{L}}{\mathrm{R}_0} &= \frac{\xi\chi_{\mathrm{L}}}{\Upsilon_i^2}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where we have written

$$\left.\begin{aligned} \chi_{\mathrm{R}} &= \mathrm{S}_v\mathrm{S}_i+\mathrm{S}_v'\mathrm{S}_i', \\ \xi\chi_{\mathrm{L}} &= \mathrm{S}_v'\mathrm{S}_i-\mathrm{S}_v\mathrm{S}_i'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (23)$$

In view of (17) we may write

$$\frac{\mathrm{L}}{\mathrm{C}_1\mathrm{R}_0^2} = \frac{2\chi_{\mathrm{L}}}{\Upsilon_i^2}.$$

(*b*) C, *r* branch :

$$r\frac{dy}{dt}+\frac{y}{\mathrm{C}} = \frac{dv}{dt},$$

$$j\left[\left(pr.\mathrm{S}_y+\frac{\mathrm{S}_y'}{\mathrm{C}}\right)\cos pt-\left(pr.\mathrm{S}_y'-\frac{\mathrm{S}_y}{\mathrm{C}}\right)\sin pt\right] = pv_0[\mathrm{S}_v\cos pt-\mathrm{S}_v'\sin pt],$$

$$r\mathrm{S}_y+\frac{1}{p\mathrm{C}}\mathrm{S}_y' = \mathrm{R}_0\mathrm{S}_v,$$

$$r\mathrm{S}_y'-\frac{1}{p\mathrm{C}}\mathrm{S}_y = \mathrm{R}_0\mathrm{S}_v',$$

$$\left.\begin{aligned} \frac{r}{\mathrm{R}_0} &= \frac{\chi_r}{\Upsilon_y^2}, \\ \frac{\mathrm{C}}{\mathrm{C}_1} &= \frac{2}{\xi}\frac{\Upsilon_y^2}{\chi_c}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (24)$$

where

$$\begin{aligned} \chi_r &= \mathrm{S}_v\mathrm{S}_y+\mathrm{S}_v'\mathrm{S}_y' \\ &= \mathrm{S}_v(1-\mathrm{S}_i)-\mathrm{S}_v'\mathrm{S}_i', \end{aligned}$$

or

$$\chi_r = \mathrm{S}_v-\chi_{\mathrm{R}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

similarly

$$\chi_c = \xi\chi_{\mathrm{L}}-\mathrm{S}_v'.$$

$$\left[\text{Check} \qquad ry^2 = \frac{v^2}{\overline{\text{R}}} - \text{R}i^2 = \frac{\hat{v}_0^2 \text{S}_v}{\text{R}_0} - \hat{j}_0^2 \text{R}_0 \chi_\text{R}\right.$$

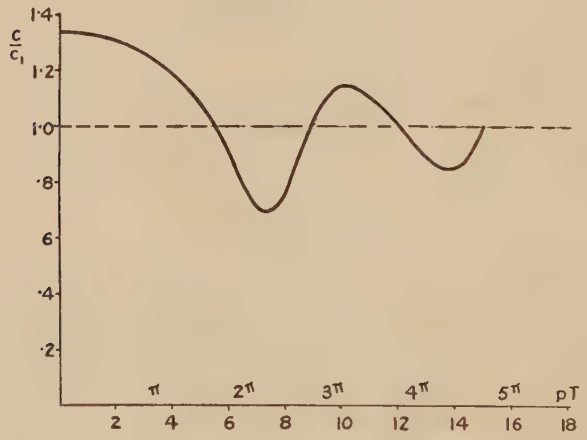
$$\left. \therefore \quad \frac{r}{\text{R}_0} = \frac{\text{S}_v - \chi_\text{R}}{\Upsilon_y^2 \xi}. \right]$$

The values of

$$\frac{\text{R}}{\text{R}_0}, \quad \frac{r}{\text{R}_0}; \quad \frac{\text{L}}{\text{C}_1 \text{R}_0^2}; \quad \frac{\text{C}}{\text{C}_1}$$

are plotted on figs. 6, 7, 5*b* for values of ξ up to 4π or over *. It is not possible to publish full tables for checking these curves, but the most important quantities will be found

Fig. 5*b*.



in Table III., which gives full details for calculating the S and Υ functions. The quantities S_i, S_i', S_v, S_v', Υ_i^2, Υ_y^2 will be found plotted on figs. 8 and 9.

The fact that the principal resistance R of the equivalent circuit not only assumes negative values but has negative maxima larger than its positive maxima is very significant. This unexpected result is hard to explain in any language

* From fig. 7 we note that L goes negative for the same range of ξ values as does the conductance (fig. 4). Qualitatively the case is the same as if we had applied a "j^2-transformation" (van der Pol, 'A New Transformation in A.C. Theory,' Proc. I. R. E. xviii. 2, p. 221, Feb. 1930).

Quantitatively the case is obviously more complicated; moreover, the capacity never goes negative, though it tends to lower values over the regions concerned. I believe this is the first case of negative inductance turning up in a physical problem.

[*To face page* 476.

TABLE III.

1. ξ.	2. ξ^2.	3. ξ^3.	4. $\frac{2}{\xi^2}$	5. $\frac{12}{\xi^3}$	6. $\sin\xi$.	7 $\cos\xi$.	8. $\xi\sin\xi$.	9. $\xi\cos\xi$.	10. S_1.	11. S_1'.	12. S_0'.	13. S_0.	14. Y_1^2	15. Y_0^2.
0	0	0	∞	∞	·000000	1·000000	·000000	·000000	1·000000	·000000	·000000	1·000000	1·00000	1·000000
0·4	0·16	0·02560	12·500...	468·750...	·389418$_3$	·921061$_0$	·155767	·368424	0·960354	−·262424	−·119240	0·989379	0·99115	0·99309
1·0	1·00	1·00000	2·00...	12·00....	·841471	·540302$_3$	·841471	·540302	0·763547	−·602337	−·288320	0·935093	0·94582	0·95753
1·4	1·96	3·84160	1·020408	3·12370	·985449$_7$	·1699671	1·379630	·237954	0·560813	−·762751	−·388551	0·875997	0·89630	0·91834
$\frac{\pi}{2}$ 1·570796	2·4674011	6·08807	0·810569	1·97107	1·000000	·0000000	1·570796	·000000	0·462670	−·810569	−·427248	0·845991	0·87109	0·89824
1·8	3·24	10·49760	0·617284	1·143118$_4$	·9738476	−·227202$_1$	1·752926	−·408964	0·324521	−·853587	−·474784	0·801872	0·83392	0·86842
$\frac{3\pi}{4}$ 2·356195	5·55165$_2$	30·82083	0·360288$_1$	0·389347	·707106$_8$	−·707106$_8$	1·666081	−1·666081	−0·014780	−·854948	−·566900	0·680630	0·73114	0·78123
2·8	7·84	61·46560	0·255102	0·195281	·334988$_{15}$	−·942222$_5$	0·937967	−2·638222	−0·256187	−·758472	−·615070	0·575244	0·64089	0·70918
π 3·141593	9·869604$_4$	97·40909	0·202642$_1$	0·123192	·0000000	−1·000000	0·000000	−3·141593	−0·405285	−·636620	−·636620	0·492768	0·56954	0·64810
3·6	12·96	167·9616	0·154321	0·0714449	−·442520$_4$	−0·896758$_4$	−1·593073	−3·228330	−0·538554	−·429909	−·645341	0·384844	0·47486	0·56457
4·0	16	256	0·125	0·0468750	−·756802$_5$	−0·653643$_6$	−3·027210	−2·614574	−0·585107	−·232221	−·635892	0·296930	0·39628	0·49253
$\frac{3\pi}{2}$ 4·71239	22·20662	493·1335	0·090063	0·0243342	−1·0000000	−0·0000000	−4·712390	−0·000000	−0·514477	+·090063	−·587754	0·163340	0·27280	0·37213
5·2	27·04	731·1616	0·07396$_45$	0·01640295	−·8834547	+0·468516$_7$	−4·593964	+2·436287	−0·379101	+·245543	−·538847	0·092785	0·20401	0·29896
5·6	31·36	983·4496	0·063775$_6$	0·01220195	−·6312666	+0·7755659	−3·535093	+4·343169	−0·239766	+·317247	−·493874	0·048612	0·15813	0·24627
2π 6·283186	39·47842	1558·5455	0·0506606	0·0$_2$7699486	·000000	1·0000000	0·000000	+6·283185	0·000000	+·318310	−·415065	0·000000	0·10132	0·17228
6·8	46·24	2138·1376	0·0432526	0·0$_2$561256	·494113$_3$	0·8693975	3·359970	5·911903	+0·139679	+·234334	−·359815	−·017391	0·07442	0·12978
7·2	51·84	2687·3856	0·0385802	0·0$_2$446531	·7936679	0·608351$_3$	5·714410	4·380129	+0·205353	+·138405	−·322399	−·022019	0·06133	0·10443
$5\pi/2$	61·68503	3805·043	0·032424$_2$	0·0$_2$315371	1·0000000	0·0000000	7·853982	0·000000	+0·222225	−·032423	−·273109	−·018462	0·04949	0·07493
8·4	70·56	4978·7136	0·0283447	0·0$_2$241026	·854598$_9$	−0·519288$_7$	7·178631	−4·362025	+0·160412	−·147364	−·243708	−·009979	0·04760	0·05949
9·0	81	6561·0000	0·0246914	0·0$_2$182399	·412118$_5$	−0·911130$_3$	3·709066	−8·200172	+0·044394	−·212649	−·222267	−·000206	0·04719	0·04940
3π	88·82644	7890·1364	0·0225158	0·0$_2$152089	·0000000	−1·0000000	0·0000000	−9·424778	−0·045032	−·212208	−·212207	+·006084	0·04706	0·04507
10	100	10000	0·02	0·0$_2$12	−·544021$_1$	−0·839071$_5$	−5·440211	−8·390715	−0·145586	−·156934	−·203237	+·010942	0·04582	0·04141
$7\pi/2$	120·90165	14617·21	0·0165423$_7$	0·0$_3$820950	−1·0000000	0·0000000	−10·995574	0·000000	−0·198435	+·0165424	−·192560	+·010669	0·03965	0·03709
4π	157·91367	24936·727	0·0126651$_5$	0·0$_3$481218	0·000000	1·0000000	0·000000	12·56637	0·000000	+·159155	−·171249	0·000000	0·02533	0·02933

Fig. 6.

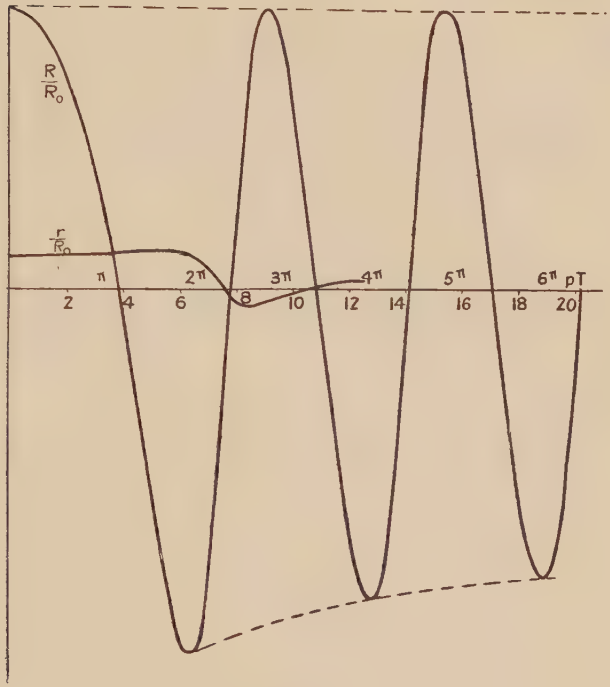
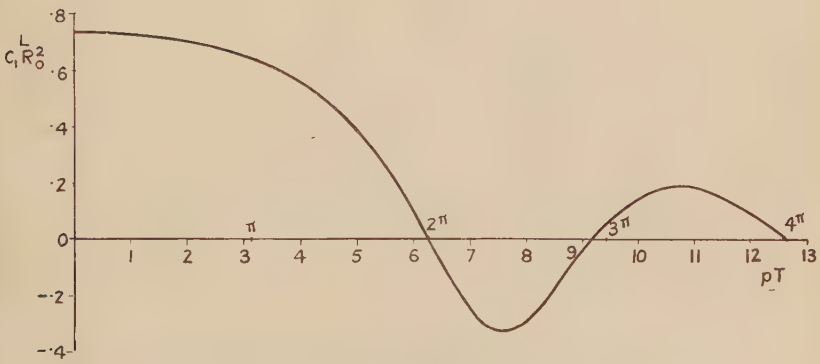


Fig. 7.



Note.—The above curve gives the inductance L in microhenries in the case of a diode of unit area for which $R_0 = 1000\,\Omega$, $C_1 = 1\,\mu\mu F$.

In general ordinates give inductance in microhenries when multiplied by $AC_1\left(\frac{R_0}{1000}\right)^2$ where A = area of plate.

Fig. 8.

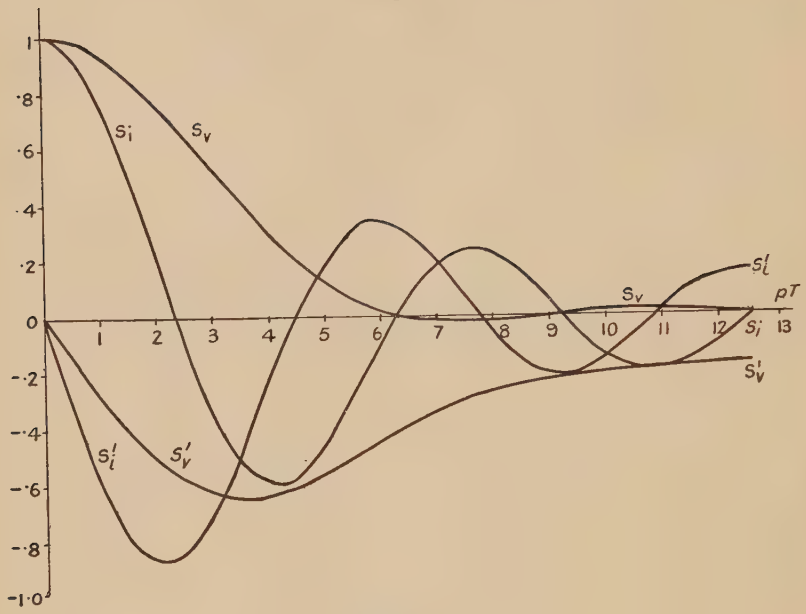
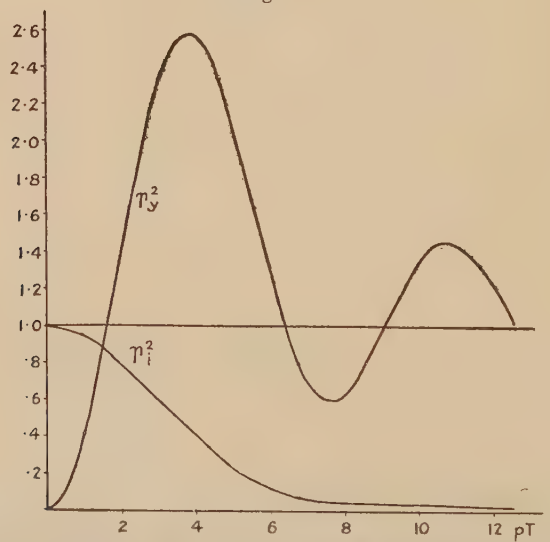


Fig. 9.



Showing how the alternating current through the system is divided up at different values of pT. Modulus of total a.c. current is indicated by a horizontal straight line.

other than mathematics, which has been purposely given in some detail.

The discussion of the part played by R in the production of oscillations is taken up in section 17.

10. Before proceeding to a discussion of these results let us first satisfy ourselves that the two resistances R and r combine to give the effective resistance $\bar{\mathrm{R}}$ (equation (13), p. 469). In order that this shall be so we must have

$$\frac{\bar{v}^2}{\bar{\mathrm{R}}} = \mathrm{R}\overline{i^2} + r\overline{y^2}$$

or

$$\frac{1}{2}\frac{\hat{v}_0^2}{\mathrm{R}_0}\Upsilon_v^2 = \frac{1}{2}\hat{j}^2\mathrm{R}_0(\chi_\mathrm{R} + \chi_r)$$

$$= \frac{1}{2}\frac{\hat{v}_0^2}{\mathrm{R}_0}\mathrm{S}_v,$$

whence

$$\frac{\bar{\mathrm{R}}}{\mathrm{R}_0} = \frac{\Upsilon_v^2}{\mathrm{S}_v},$$

in agreement with equation (13).

The relation between true and effective values can be expressed as follows:

$$\frac{1}{\bar{\mathrm{R}}} + jp\bar{\mathrm{C}} = \frac{1}{\mathrm{R}+jp\mathrm{L}} + \frac{1}{\left(r + \frac{1}{jp\mathrm{C}}\right)}.$$

If the terms on the right-hand side be rationalized separately we readily find

$$\left.\begin{aligned} \frac{1}{\bar{\mathrm{R}}} &= \frac{\mathrm{R}}{\mathrm{R}^2+p^2\mathrm{L}^2} + \frac{p^2\mathrm{C}^2 r}{1+p^2\mathrm{C}^2r^2} \\ \bar{\mathrm{C}} &= \frac{\mathrm{C}}{1+p^2\mathrm{C}^2r^2} - \frac{\mathrm{L}}{\mathrm{R}^2+p^2\mathrm{L}^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (26)$$

The above are general formulæ for effective conductance and capacity in terms of an equivalent circuit which may be extended to cover thermionic systems in general (see section (11)).

[As a check on the above expressions we note that at $\xi = 2\pi$

$$\mathrm{R} = -1{\cdot}304\,\mathrm{R}_0,$$

$$r = +0{\cdot}12\,\mathrm{R}_0,$$

$$\mathrm{L} = 0,$$

$$\mathrm{C} = {\cdot}843\,\mathrm{C}_1.$$

With the help of the formula $pC_1R_0 = \frac{1}{2}\xi$ we obtain

$$p^2C^2r = \frac{\cdot 843}{R_0},$$

$$p^2C^2r^2 = \cdot 100,$$

whence

$$\frac{R_0}{\bar{R}} = -\frac{1}{1\cdot 304} + \frac{0\cdot 843}{1\cdot 100} = 0,$$

$$\frac{\bar{C}}{C_1} = \frac{\cdot 843}{1\cdot 100} = \cdot 766,$$

as may be confirmed by reference to figs. 4 and 5*a*.

As a further check let us take $\xi = 7\cdot 2$. At this value of ξ

$$R = -0\cdot 800\, R_0,$$

$$r = +0\cdot 0402\, R_0,$$

$$L = -0\cdot 314\, C_1R_0^2,$$

$$C = 0\cdot 697\, C_1.$$

Proceeding as in the first example,

$$\frac{R_0}{\bar{R}} = \frac{\cdot 8}{\cdot 64(-\cdot 286 \times 3\cdot 6)^2} + \frac{(3\cdot 6 \times \cdot 697)^2 \times \cdot 0402}{1 + (3\cdot 6 \times \cdot 697 \times \cdot 0402)^2}$$

$$= -\frac{\cdot 8}{1\cdot 7} + \frac{\cdot 252}{1\cdot 012}$$

$$= -0\cdot 22,$$

$$\frac{\bar{C}}{C_1} = \frac{\cdot 697}{1\cdot 012} - \frac{-\cdot 286}{1\cdot 7}$$

$$= 0\cdot 857.$$

As a check on the expression for $\bar{C}$ at $p = 0$ (suffix 0),

$$\bar{C}_0 = C_0 - \frac{L_0}{R_0^2},$$

$$\therefore \quad \frac{\bar{C}_0}{C_1} = \frac{C_0}{C_1} - \frac{L_0}{C_1R_0^2},$$

in agreement with the result arrived at in section 8 and with figs. 5*a*, 5*b*, and 7.]

11. Now the diode behaves as a resistance $\bar{R}$ paralleled by a capacity $\bar{C}$ (fig. 3*a*) as far as measurements in an externally connected circuit are concerned. It may be as well, therefore, to devote some space to the justification of the analysis of the diode into the arrangement of fig. 3 *b*.

We note, firstly, that the current through the system is necessarily composed of two parts, which I have referred to as conduction current and displacement current.

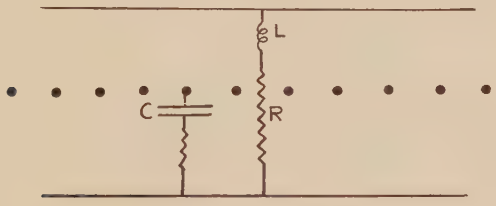
Secondly, we note that neither of the two parts into which the original equations automatically divide the currents can be made to satisfy either of the equations

$$\bar{R}i = v,$$

$$y = \bar{C}\frac{dv}{dt}.$$

Hence we can say at once that fig. 3*a* does not represent the true state of affairs. Each branch of fig. 3*b*, however, satisfies the circuit equations perfectly. By considering any more complicated network satisfying the equations it can be shown that (as far as measurements in an external circuit are concerned) the representation of conduction and displacement current paths cannot be made more general than

Fig. 10.



indicated in fig. 3*b*. It will therefore be assumed that L, R, "conduction branches," and *r*, C, "displacement branches," are a general feature of thermionic systems. The relative magnitudes of L, R, *r*, and C will, of course, depend on the electrode configuration. In multi-electrode systems each electrode will in general have an L, R branch and a *r*, C branch.

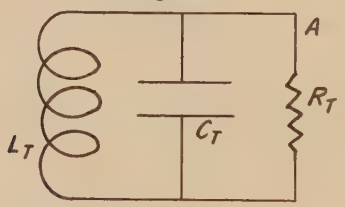
12. Direct experimental evidence as to the difference between "true" and "effective" values in the case of a diode is not forthcoming. Indirect experimental evidence is, however, obtainable. Referring to fig. 10, which represents a triode with a negative grid, we note that the L, R branch goes to the anode, the *r*, C branch to the grid. The anode has a small *r*, C branch also, but the point emphasized is that there is no L, R branch to the grid. For the grid-cathode capacity, therefore, we should expect $\bar{C} = C > C_1$. (There would be a small difference between $\bar{C}$ and C due to the

presence of r, but at technical frequencies this is negligible.) It is not to be expected that the formula $C=\frac{4}{3}C_1$ holds quantitatively in this case, as the grid upsets the space-charge conditions assumed for the diode, but we look for $C>C_1$ on the basis of the fact that the grid takes no conduction current, and is therefore "unaware" of the inertial effects of the electrons, the presence of which should raise the capacity.

That $\overline{C}>C_1$ in the case of the grid-cathode capacity of a triode, and $\overline{C}<C_1$ in the case of a diode, I have been able to prove experimentally.

13. The experiments were based on an important property of circuits established theoretically by Doborzynski *. Referring to fig. 11, a current-indicating device at A serves to measure the current flowing through a resistance R_T in

Fig. 11.



parallel with a capacity C_T and an inductance L_T. (The only object of inserting a suffix is to distinguish tuning inductance and capacity from valve inductance and capacity for which the symbols L, C are already in use.)

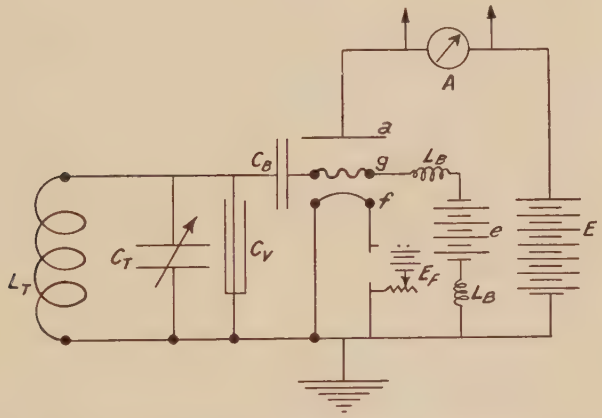
The coil L_T is coupled to an an oscillator (not shown). The capacity C_T may be varied. Doborzynski showed that the value of C_T for maximum current in A is independent of the value of R_T. It is absolutely essential that A should only measure current which has passed through R_T.

The adaptation of the circuit of fig. 11 to the measurement of valve capacities needs careful explanation. We can, of course, replace C_T and R_T by a thermionic system (*cf.* fig. 3*a*), but we cannot in that case insert A in series with $\overline{R}$ alone. A must of necessity be external to the thermionic system. In order to get over this difficulty I decided to take advantage of the rectifying property of thermionic systems, and to tune by means of the rectified component of current. Since

* *Zeit. für Hochfrequenztechnik*, xxxi. pp. 15–17 (Jan. 1928).

the rectified component of current is direct current, the condition that A should only measure current which has passed through $\overline{R}$ is satisfied. It should be noted that the alternating component of current passing through $\overline{R}$ is assumed to attain a maximum value at the value of tuning capacity for which the rectification effect is also a maximum. This assumption can, however, be immediately justified. Unlike a capacity of an inductance a resistance necessarily has a maximum voltage across it when the current passing through it is a maximum. Thus our d.c. indicating instrument tells us when there is a maximum voltage across the resistance branch * of the thermionic system, *i.e.* when the current (a.c. as well as d.c.) is a maximum.

Fig. 12.



The circuit adopted for experimental work is shown in fig. 12, from which the oscillator has been omitted. Oscillations are induced in the circuit of fig. 12 by loosely coupling the coil L_T to the oscillator. The frequency of the oscillator was in the neighbourhood of 3×10^6 cycles. Such a frequency was decided upon as the best compromise between the two requirements, (*a*) sensitivity, and (*b*) an absence of effects due to the magnitude of pT. With regard to (*b*), it will be noticed that pT variations become of importance only for values of pT greater than 1, or even 2, for capacity measurements. This may be seen by reference to figs. 5*a* and 5*b*. The only case where pT will

* The effect of the inductance L is here neglected, but this is justified at the frequencies employed.

exceed 2 in practice will be when very low space currents are taken. Fortunately, however, this leads to very little error, as when the space current is very low the space-charge is very highly rarefied except for a very small region just in front of the cathode*, so that we should expect the dielectric constant to approach unity in this case, as measurements actually confirm. Referring again to fig. 12, C_v represents a precision vernier condenser of cylindrical shape of maximum capacity $50\mu\mu$F. C_B is a blocking condenser of negligible impedance. The plate circuit contains the microammeter A paralleled by a device (indicated by arrows) for balancing out the steady plate current through the valve. Chokes, L_B, were placed in the grid circuit, one on either side of the grid battery E_c. Since the external output resistance is negligible compared with the valve resistance, the capacity under observation is in the triode case $C_{gf}+C_{ga}$†. In the experiments where the valve was used as a diode the grid battery was removed and the anode connected to the grid. Both ends of the filament were maintained at h.f. earth potential, as were one end of L_T, C_T, and C_V. The whole circuit was arranged as compactly as possible on a wooden table. The stability of the circuit was found to be such that a layer of copper gauze placed under the apparatus and earthed effected no improvement. Hand capacity to the apparatus was eliminated by the extension handle to the precision condenser.

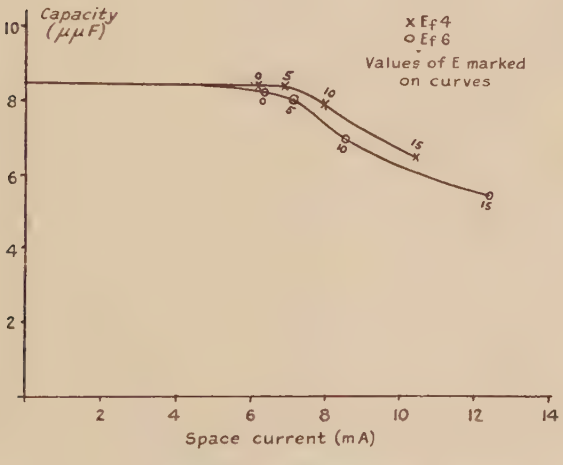
14. The experimental procedure is straightforward. In the triode case E_b was kept constant and E_c varied, thus varying the space current and so the interelectrode capacities. For each setting of E_c the value of C_V for maximum rectification is noted. The balanced microammeter A is a sufficiently accurate indicator, even over the "straight" part of the characteristic, in the case of a triode. It was occasionally necessary to determine the actual position of the maximum by taking a pair of values of C_V to give rectified currents one on either side of the maximum The fact that even this refinement was not as a rule effected indicates how sharp is the resonance curve of rectified current in cases where the damping of the input circuit is small. Separate experiments to determine cold capacities and corrections for supports and lead in wire capacities were made by plugging in the valve itself in one case, and a "dummy" valve in the other, the dummy valve having had

* *Cf.* Rakshit, Phil. Mag. ix. 55, p. 80 (Jan. 1930).
† *Cf.* van der Bijl, 'Thermionic Vacuum Tube,' p. 208.

its anode and grid clipped off at the points where they are welded to the support wires.

In the case of a diode the method is very much less sensitive. It will be realized that the parallel conductance path (sensibly absent in the triode case) introduces serious damping in the diode case. Also at high space currents the rectification effect is smaller. It was accordingly necessary to increase the oscillatory input voltage amplitude from about 1 volt to 5 volts r.m.s. The depression of dielectric constant was found to vary with the input voltage, being greater the greater the input voltage. The variation of $\frac{C}{C_1}$

Fig. 13.



N.B.—Deduct 2·5 $\mu\mu$ F extraneous capacities.

with input voltage could not, however, be studied profitably with the apparatus available at the time. The results, using an r.m.s. input voltage of 5, are quoted as constituting an important piece of evidence, *e.g.*, that dielectric constant is less than unity in the case of a diode, the depression of dielectric constant being greater the greater the space current flowing. Fig. 13 shows the result of experiments on an L.S. 5B valve used as diode. Fig. 14 shows that the dielectric constant is greater than unity in the case of the same L.S. 5B valve used as triode. Fig. 15*a* shows two anode characteristics taken on a Marconi D.E.P. 215 valve. Fig. 15*b* shows that the ratio of hot to cold capacity follows

Fig. 14.

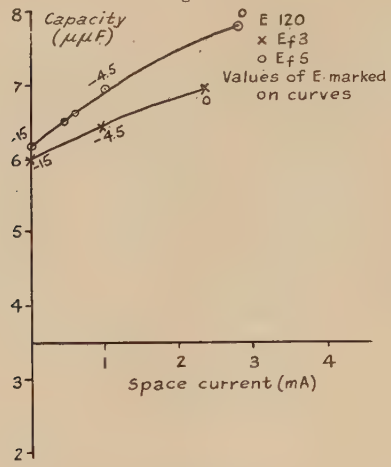
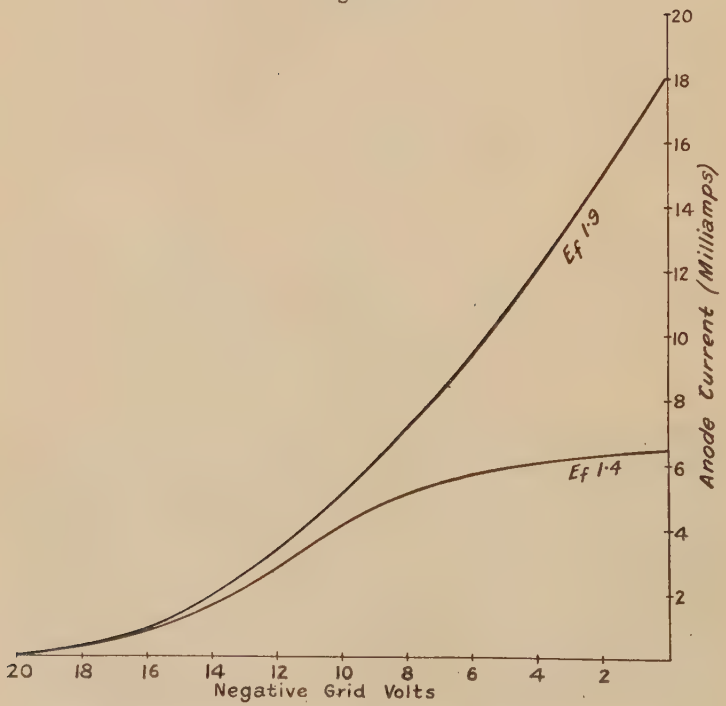


Fig. 15 *a*.

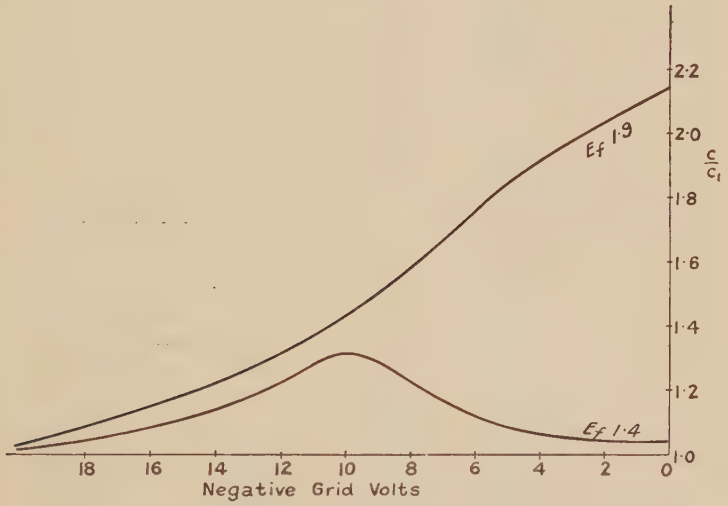


Anode characteristics of Marconi D.E.P. 215 type valve used for $\frac{C}{C_1}$ measurements. Anode volts 120.

the anode characteristic slope very closely. The results on a low impedance valve, such as the D.E.P. 215, are more pronounced than those for a high impedance valve, such as the L.S.5B as would be expected from the space currents taken in the two cases.

15. A detailed interpretation of the results will not be attempted. It will be clear that the ideal theoretical value (3/5 or 4/3 as the case may be) is only intended to apply to a particular value of space current, or, rather, to a particular distribution of space-charge, which is difficult to

Fig. 15*b*.



Experimental input capacity measurements corresponding to anode characteristics shown on fig. 15*a*.

approach in practice owing to the effect of initial velocities. Also, the valves used do not closely resemble a parallel plane structure, and the values of $\frac{\overline{C}}{\overline{C}_1}$, $\frac{C}{C_1}$ are always greater in the cylindrical than in the plane case (*cf.* Table VII., section 19). It is worth while commenting on the fact that the ratio $\frac{C}{C_1}$ appears to follow the anode characteristic slope in the case of a triode. C, C_1 here refer, of course, to

values of the effective input capacity. The experimental results suggest an empirical formula of the type

$$\frac{C}{C_1} = 1 + Ak_p,$$

where k_p is the anode characteristic slope. It was thought for some time that this was the best approach to the experi-

Summary of Input Capacity Experiments on D.E.P. 215.

TABLE IV.*a*.—E_f 1·9, E 120.

$-e$.	R_p.	C/C_1.	$1-C_1/C$.	$R_p(1-C_1/C.)$.
15	$11{\cdot}76\times10^3$	1·183	0·156	$1{\cdot}84\times10^3$
12	7·84	1·318	0·240	1·88
10	6·45	1·429	0·300	1·94
8	5·67	1·579	0·367	2·08
6	5·00	1·759	0·432	2·16
4	4·42	1·911	0·477	2·10
2	4·00	2·030	0·507	2·02
0	3·74	2·145	0·533	2·00

TABLE IV.*b*.—E_f 1·4, E 120.

$-e$.	R_p.	C/C_1.	$1-C_1/C$.	$R_p(1-C_1/C)$.
15	$14{\cdot}65\times10^3$	1·110	0.100	$1{\cdot}47\times10^3$
12	9·46	1·225	0·184	1·74
11	8·57	1·280	0·218	1·87
10	9·81	1·312	0·238	2·34
8	14·64	1·230	0·188	2·75
6	23·9	1·117	0·106	2·54
4	45·5	1·060	0·057	2·59
2	63·2	1·041	0·040	2·53
0	63·2	1·039	0·038	2·40

mental results, but I have recently noticed that a much better empirical formula is the following :

$$\frac{C_1}{C} = 1 - Ak_p$$

$$= 1 - \frac{B}{R_p}, \text{ where } B = A\mu.$$

Tables IV.*a* and IV.*b* indicate the constancy of B. The

values of R_p are derived from very careful slope measurements on the curves of fig. 15. The value of μ is taken as constant and equal to 6. Actually μ varies slightly over the characteristic, as may be confirmed in any triode by slopemeter methods. Subsequent measurements reveal that in the case of a D.E.P. 215 the μ variations are nearly sufficient to account for the variations in B recorded in the last column of each table (*cf.* Appendix II.).

It must be pointed out that the time devoted to experiments on valve capacities was less than might have been supposed from the space devoted to the subject, the whole of the experimental work having taken place between 26.vi.28 and 13.vii.28. Generally speaking, it may be said that the method shows promise as a rapid and convenient method for measuring the interelectrode capacities of triodes and of high impedance diodes. The use of Doborzynski's circuit renders it unnecessary to know the value of the input admittance of the thermionic system. Since the above experiments were made, Hartshorn * has described a bridge method for valve capacities and conductances, using a telephonic frequency oscillator. My experimental results on triodes, as far as they go, agree well with those of Hartshorn as far as *they* go. Hartshorn did not investigate the effect of varying grid volts; I did not investigate the effect of varying frequency. It should be noted that the variations of $\frac{c}{c_1}$ with frequency observed by Hartshorn *at telephonic frequencies* cannot on my theory be attributed to purely electronic space-charge. I can only suggest that the variations observed by Hartshorn are due either to the dielectric of the valve bases or to that of the space-charge *modified by the presence of residual gaseous ions* (see next section) †.

16. I have used the term "dielectric constant" in relation to space-charge perhaps more freely than would be justified without further qualification, which I now give. We have seen that, provided no current passes between the electrodes used for measurement, an increase in capacity is recorded. The same electrons which increase the capacity between cathode and negative grid can decrease the capacity between cathode and positive anode. The "dielectric constant" of the space-charge constituted by a given cloud of

* *Loc. cit.*

† *Note added in proof.*—The reader is referred to the admirable work of Appleton and Childs, Phil. Mag. [10] lxvii. p. 969, Dec. 1930, "On some Radio-Frequency Properties of Ionized Air."

electrons has then a range of values varying from some value greater than 1 characteristic of zero conductance to some value less than 1 determined by the degree of conductance taking place. If the anode is only slightly conducting the dielectric constant may, of course, exceed unity, but will never exceed the value obtained for zero conductance.

It is interesting to note that the amplification factor μ of a triode is intimately dependent on the dielectric constant of space-charge, and varies sometimes slightly, sometimes considerably, over the anode characteristic, as may be confirmed experimentally by means of a suitable bridge. The variations in μ are worthy of attention from those engaged in valve design, as if μ departs too much from the generally accepted constant value distortion results. A very approximate treatment of the case of μ variation arising from variations in space-charge conditions is given in Appendix II.

It is important to compare and contrast the dielectric constant of space-charge with that of a layer of electrons in loose association with gaseous ions*. In the latter case the well-known formula for the depression of dielectric constant is

$$\delta\epsilon = -\frac{4\pi \mathrm{N}e^2}{mp^2}. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The kind of difference we are concerned with is brought out very clearly in a paper by Sven Benner entitled "The Change in the Dielectric Constant of a very Rarefied Gas by Electrons"†. Benner points out that in deriving the formula (27) the electrons are supposed to start from rest after every collision with an ion. In case the gas is sufficiently rarefied so that the mean free path greatly exceeds the distance travelled by electrons under the influence of an alternating field between infinite condenser plates, Benner finds for the conductivity and depression of dielectric constant

$$\left.\begin{aligned} \sigma &= \frac{\mathrm{N}e^2}{mp^2\mathrm{T}}(1-\cos p\mathrm{T}), \\ \delta\epsilon &= -\frac{4\pi \mathrm{N}e^2}{mp^2}\left(1-\frac{\sin p\mathrm{T}}{p\mathrm{T}}\right). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (28)$$

We shall interest ourselves only with Benner's expression for $\delta\epsilon$. When much gas is present T may be regarded as very large, and the formula reduces to (27) (except at very

* Eccles, Proc. Roy. Soc. A, lxxxvii. p. 79 (1912); Larmor, Phil. Mag. xlviii. p. 1025 (1924); and others.

† *Ann. der Phys.* 3, vii. p. 993 (1929).

low values of p). When no gas is present, pT is small up to quite large values of p. The formula reduces in this case to

$$\delta\epsilon = -\frac{2\pi N e^2}{3m} T^2, \quad \ldots \quad (29)$$

a result which, like my own, is independent of p for pT small. It is interesting to apply (29) to the case where N and T have values characteristic of the space-charge case. We then find

$$\left[\text{since } 4\pi N e = 4\pi P = \frac{2m}{9e} a^2 x^{-2/3} \right.$$

$$\text{and} \qquad \left. T^2 = \frac{9}{a^2} x^{2/3} \right]$$

$$\delta\epsilon = -\tfrac{1}{3}. \quad \ldots \quad (30)$$

Thus the application of Benner's formula to each point of a thermionic system in which mutual repulsions determine a certain non-uniform distribution, yields a value 2/3 for the dielectric constant, differing by 11 per cent. from the value 3/5 obtained in the space-charge case having that non-uniform distribution.

The difference between Benner's formula and mine lies in the fact that Benner overlooks the space distribution of N which occurs when the gaseous ions are removed. In this connexion it is readily shown that in the purely electronic case mutual repulsions are always of importance in cases where N is sufficient to make $\delta\epsilon$ appreciable.

Thus, when pT is made large we can only use Benner's formula (28) on the understanding that pT is made large by introducing gaseous ions (in the presence of which the distribution of N is sensibly uniform), thus increasing the effective value of T and yielding in the limiting case the formula (27).

My formula

$$\epsilon = \frac{2}{p\mathrm{T}} \frac{\mathrm{S}'_v}{\Upsilon_v^2} \quad \ldots \quad (31)$$

must be used in the purely electronic case, especially for large values of pT. (31) may be written in the form

$$\left.\begin{aligned} \delta\epsilon = \frac{1}{\cdot} - \epsilon = \frac{4\pi N e^2}{m p^2} f(\xi), \\ \text{where} \qquad f(\xi) = \frac{1}{2}\xi^2\left(1 + \frac{2\mathrm{S}'_v}{\xi \Upsilon_v^2}\right), \\ \xi = p\mathrm{T}. \end{aligned}\right\} \quad \ldots \quad (32)$$

[N.B.—N=no. of electrons per c.c. at place where time of transit=T.]

Comparison between Benner's factor $\left(1-\frac{\sin\xi}{\xi}\right)$ and $f(\xi)$ leads to widely different values at large values of ξ. Even at small values of ξ the formulæ

$$1-\frac{\sin\xi}{\xi}=\frac{1}{6}\left(1-\frac{\xi^2}{20}\right)$$

and

$$f(\xi)=\frac{1}{5}\xi^2\left(1-\frac{69}{12600}\xi^2\right)$$

TABLE V.

ξ.	$1-\frac{\sin\xi}{\xi}$.	$f(\xi)$.
0	0	0
0·4	·0265	·032
1	·158	·199
1·8	·459	·637
2·8	·880	1·497
4	1·189	2·832
$3\pi/2$	1·212	3·685
2π	1·000	4·601
$5\pi/2$	0·873	2·252
8·4	0·898	0·847
9	0·954	0·000
3π	1·000	0·001
10	1·054	0·800
4π	1·000	5·597
5π	1·000	0·000
6π	1·000	5·920

depart from one another rapidly as ξ increases above about 0·4. In studying Table V. the fact that $\frac{4\pi Ne^2}{mp^2}$ becomes small when p is large must not be forgotten.

One conclusion which may be drawn from Table V. is that the amplitude of the oscillations in which the ϵ, ξ curve for electrons between infinite parallel planes approaches unity (*cf.* fig. 5 *a*) is too small when Benner's formula is used.

17. The equivalent impedance of a diode according to fig. 3 *b* is

$$\frac{\frac{\mathrm{L}}{\mathrm{C}}+\mathrm{R}r+j\left(rp\mathrm{L}-\frac{\mathrm{R}}{p\mathrm{C}}\right)}{\mathrm{R}+r+j\left(p\mathrm{L}-\frac{1}{p\mathrm{C}}\right)}.$$

It is not difficult to show that the impedance as given above is always capacitative, as would, of course, be expected from figs. 2 and 5 *a*.

We have seen that, as far as the diode is concerned, the resistances R, r of fig. 3 *b* are equivalent to the effective resistance R of fig. 3 *a*. If σ denotes the conductance of the system *expressed as a fraction of its value at zero frequency* and W the corresponding quantity for mean power input, we see from equations (12) and (15)

$$\mathrm{W}=\sigma=\frac{\mathrm{S}_v}{\mathrm{T}_v{}^2}.$$

Thus, when σ is negative (fig. 4), W will be negative also. It is convenient to denote negative conductance by a separate symbol, as follows :

$$\sigma_-=-\sigma.$$

As is well known, the inductance capacity and resistance of any leads connected with the valve must be taken into account. Let us imagine the diode $\overline{\mathrm{C}}$, $\overline{\mathrm{R}}$ to be associated with the external circuit $\mathrm{L_T}$, $\mathrm{C_T}$, $\mathrm{R_T}$, as shown in fig. 16. Then it can be shown* that oscillations of angular frequency

$$\left[\left(1-\frac{\mathrm{R_T}}{\mathrm{R}_0}\sigma_-\right)\Big/\mathrm{L_T}\overline{\mathrm{C}}'\right]^{1/2} \quad . \quad . \quad . \quad . \quad (32)$$

will be sustained in the circuit of fig. 16 provided that at this frequency

$$\frac{\mathrm{L_T}}{\overline{\mathrm{C}}'\mathrm{R_T}}>\frac{\mathrm{R}_0}{\sigma_-}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where

$\overline{\mathrm{C}}'=\mathrm{C_T}+\overline{\mathrm{C}}$,

R_0 = zero frequency impedance of diode.

The expression (32) will be real provided $\frac{\mathrm{R_T}}{\mathrm{R}_0}\sigma_-<1$.

* See Appendix III., where a slightly more complicated circuit is considered.

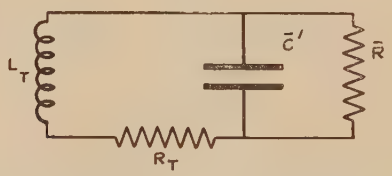
Since σ_- is never greater than 0·25 *, this condition is readily satisfied, though, owing to difficulties in reducing losses in practice, σ_- might have to have some value less than 0·25, *i. e.*, one might have to sacrifice some negative conductance in order to obtain oscillations.

The inequality (33) is likely to be satisfied in practice. Thus, taking 50 ohms as a lower limit to the static impedance of the diode (a limit which can be attained in practice using an equipotential cathode and a cathode anode clearance of 0·3 mm.), the value of $\frac{R_0}{\sigma_-}$ would be only 200. To attain

$$\frac{L_T}{\bar{C}' R_T} > 200$$

should be possible even at the highest frequencies with which we are concerned. We can calculate the frequency corresponding to $R_0 = 50$, $\sigma_- = 0{\cdot}25$, since we know the

Fig. 16.



Diode and associated circuit in simple case (diode capacity $\bar{C}$ assumed in parallel with extraneous capacity).

value of ξ corresponding to $\sigma_-0{\cdot}25$, and T may be calculated from equation (18), in which C_1 is obtained by taking plates of (say) unit area 0·3 mm. apart. We thus obtain

$$pT = 7{\cdot}7,$$

$$T = 2{\cdot}95 \times 10^{-10}\ \dagger,$$

whence $$n = 4{\cdot}15 \times 10^9 \text{ cycles per second,}$$

which corresponds to λ 7·2 cm. There are further conditions on the external circuit in view of the fact that (32) is equal to the angular frequency p.

* Analysis shows that in the neighbourhood of $\sigma_- = 0{\cdot}25$ a 1 per cent. decrease in r (fig. 6) at const. pT would give rise to a 10 per cent. increase in σ_-. If we could eliminate r completely σ_- would be ·767 at $pT = 2\pi$. Unfortunately, r cannot apparently be independently controlled.

† The time taken by e.m. waves to travel from cathode to anode is only 10^{-12} sec. (taking $\epsilon = 1$).

Squaring and re-arranging, we have

$$p^2 L_T \bar{C}' = \left(1 - \frac{R_T}{R_0}\sigma_-\right).$$

If we write

$$L_T \bar{C}' = \frac{1}{\omega^2},$$

the equation becomes

$$\frac{p^2}{\omega^2} = \left(1 - \frac{R_T}{R_0}\sigma_-\right) \quad . \quad . \quad . \quad . \quad (32\,a)$$

In the case considered above $\frac{\sigma_-}{R_0} = \frac{1}{200}$. Thus $p/\omega = 0$ when $R_T = 200$ ohms (the maximum value of R_T consistent with (32)), when $R_T = 150$ ohms $\frac{p}{\omega} = \frac{1}{2}$, and when $R_T = 100$ ohms $\frac{p}{\omega} = \sqrt{\frac{1}{2}}$. *Thus in general the valve and associated circuit must be tuned to a frequency higher than that for which strongest oscillations are required.* Since in the example chosen the diode and associated circuit would have to be tuned to $\lambda < 7{\cdot}2$ cm., we readily understand why normal diodes do not oscillate spontaneously. If, however, T could be *artificially* increased (*i. e.*, without diminution of anode potential and consequently input power) the lower limit of λ would be proportionately raised and we might hope for oscillations. The fact that a normal diode cannot in practice be made to oscillate does not affect these formulæ, which are applicable alike to normal diodes and to diodes in which the time of transit has been artificially increased, a feature which may be brought about in three ways :—

1. By the use of a perforated anode *.
2. By the introduction of gas †.
3. By the application of a magnetic field the lines of force of which are parallel to the filament ‡.

I would suggest that it is in the satisfaction of the conditions (32), (32 *a*), and (33) §, made possible by an artificial

* Gill and Morell, *loc. cit.*

† *Cf.* Knipping, *Zeit. f. Hochfrequenztechnik*, 34, i. p. 9 (July 1929).

‡ Breit. J. Opt. Soc. Am. ix. p. 709 (1924); Okabe, Proc. I. R. E. (April 1929).

§ Or similar conditions.

increase in T that the explanation of ultra-short wave oscillations of most types is to be found (in method 1 the perforated anode is frequently the grid of a triode the plate of which, if present, affects T and the space-charge distribution, but not the main argument). Of the three methods, the magnetron method (3) is, in one sense, the most efficient, the wave-length attainable * being, to the best of my knowledge, the shortest wave-length of undamped oscillation known.

It is interesting to note in passing that oscillations obtained by method (1) are known which do not obey the B.K. relation $\lambda^2 V = \text{constant}$ †. A possible explanation is substantially that put forward by Hollmann, namely, that the alteration of V so alters the space-charge distribution and the value of T for strongest oscillation (T_0) as to cause the wave-length to decrease as V is decreased. This explanation is substantiated by Table VI., in which T repre-

Table VI.

Author.	T_0.
Barkhausen, Kurz	2T
Gill	$\frac{8}{3}T_2$
Sahanek ‡	$\frac{5}{3}T$
Benham	0·82T

sents the time of transit of electrons between cathode and anode, T_2 that between grid and anode. My value is the only one for which space-charge is taken into account. The table aims at no completeness, and is intended mainly to bring out the difference in the results obtained neglecting and including space-charge.

It is thought that (allowing for the slightly different values of T resulting from the neglect of space-charge) the original observations of B. and K., using a high filament temperature, are sufficiently accounted for by my value of T_0.

In previous treatments of ultra-short wave oscillations there appears an increasing tendency to talk about space-charge, oscillating space-charge and so forth, accompanied

* λ 5·6 cm., Okabe. *loc. cit.*

† Hollmann, *Zeit. f. Hochfrequenztechnik.* 35, xxi. (Jan. 1930).

‡ *Phys. Zeit.* xxvi. p. 368 (June 5th, 1925).

by a complete omission of space-charge from the mathematics. Thus Möller * derives in section 6 of his paper an oscillating choke equivalent to an oscillating space-charge by mathematics which leaves space-charge effects out of account. That Möller and others obtain from their theories conclusions which are confirmed by experiment only shows how much may frequently be explained by theories which by common consent omit to take into account some complicating feature. Another example of this is the omission of the initial velocity distribution.

The fact which emerges when space-charge is taken into account is that whenever there is a transit of electrons between a space-charge limited cathode and an anode there is a negative resistance within the system for some value of pT. The negative resistance property may thus be regarded as inherent in the space-charge itself. The hypothesis that in the case of Barkhausen-Kurz oscillations electrons actually oscillate about the grid is thus not essential to the explanation of the oscillations, and should therefore be discarded unless it can be proved that such oscillations do actually occur. In this connexion it is significant that Kapzov † has shown on reasonable assumptions that only those electrons emitted from the filament during a small portion of a cycle will execute oscillations about the grid. Such a conclusion would appear to be inconsistent with the idea that the valve oscillations are necessarily accompanied by electron oscillations. In all probability the great majority of cases are explicable in terms of the time of transit of electrons which miss the grid one or twice only, *i. e.*, the electron "oscillations" are very short-lived, and should not in my opinion be regarded as oscillations ‡, as confusion with valve oscillations, of which electron oscillations are not an essential feature, is otherwise bound to arise, and furthermore the turning back of the electrons between anode and grid is a state of affairs which is determined in any case by the d.c. potentials of the valve electrodes. The importance of the fact that the electrons may miss the grid once or even twice lies in the magnitude of the phase difference between current and potential which would exist if an alternating potential were superimposed on the d.c. potential between cathode and grid. Since the space-charge distribution between cathode and grid is not seriously disturbed (except possibly when the oscillations

* *Zeit. f. Hochfrequenztechnik* (Dec. 1929).
† *Zeit. f. Phys.* xlix. p. 395 (1928).
‡ The word "orbit' is suggested as appropriate.

build up), the theory given for a space-charge limited diode may be taken to apply approximately to the case where the anode is perforated, provided the appropriate value of T is used.

18. It is interesting to inquire into the effect of neglecting space-charge, *i. e.*, to treat the case of a single electron under the influence of a field which will now be independent of x. We require the solution to a single equation of the form

$$\mathrm{U}\frac{\partial \mathrm{U}}{\partial x}+\frac{\partial \mathrm{U}}{\partial t}=f(t).$$

We have as our first order equation

$$\frac{\partial u}{\partial t}+\mathrm{U}\frac{\partial u}{\partial x}+u\frac{\partial \mathrm{U}}{\partial x}=f\sin pt, \quad . \quad . \quad . \quad (35)$$

where f is a constant acceleration given by

$$f=\frac{v}{\mathrm{V}}\left(\mathrm{U}\frac{d\mathrm{U}}{dx}\right).$$

If we write

$$\mathrm{F}=\mathrm{U}\frac{d\mathrm{U}}{dx},$$

we have

$$\mathrm{U}=\sqrt{2\mathrm{F}x}$$
$$=\mathrm{FT},$$
$$\frac{d\mathrm{U}}{dx}=\frac{1}{\mathrm{T}};$$

(35) becomes

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial \mathrm{T}}+\frac{u}{\mathrm{T}}=f\sin pt,$$

which, after changing variables according to the scheme,

$$t=t'+\mathrm{T},$$
$$\mathrm{T}=\mathrm{T}', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

reduces to

$$\frac{du}{d\mathrm{T}}+\frac{u}{\mathrm{T}}=f(\cos p\mathrm{T}\sin pt'+\sin p\mathrm{T}\cos pt'). \quad . \quad (37)$$

(37) has as solution, after putting $\mathrm{T}=\frac{\xi}{p}$,

$$u=\frac{f}{p}\xi(\mathrm{S}_u\sin pt'+\mathrm{S}_u'\cos pt'),$$

where

$$S_u = \frac{2}{\xi^2}(1-\cos\xi),$$

$$S_u' = \frac{2}{\xi^2}(\sin\xi - \xi).$$

Thus

$$\Upsilon_u^2 = \frac{4}{\xi^4}(2-2\cos\xi+\xi^2-2\xi\sin\xi). \quad . \quad . \quad (38)$$

The above expression is also the value of Υ_i^2 in this case, so that we see that the values of Υ_i^2 are the same whether or no space-charge is taken into account (*cf.* equation (10)). The energy expenditure in this case must have the same sign as S_u, *i.e.*, is always positive.

It is readily proved that the variation with ξ of the r.m.s. value of u' is the same as that of $\overline{u^2}$. The equation involved is

$$\frac{\overline{du'}}{d\xi} + \frac{\overline{u}}{U}\frac{\overline{du}}{d\xi} + \frac{\overline{u'}}{\xi} = 0, \quad . \quad . \quad . \quad . \quad (39)$$

in which u is already known. Thus, in the case of voltage saturation, the frequency variation of the rectification effect (in this case the same as u') is in accordance with (38), which, when expanded in powers of ξ, gives the result

$$\Upsilon_{\Delta I} = 1 - \frac{\xi^2}{18} + \frac{\xi^4}{720} - \frac{\xi^6}{50{,}400}. \quad . \quad . \quad . \quad (40)$$

(40) applies equally well to cylinders, using the appropriate value of T. The fact that in the space-charge limited case also

$$\Upsilon_{u'} = \Upsilon_u^2$$

was overlooked in Part I. owing to the error in equation (14). In the space-charge case we do not, however, find $\Upsilon_{i'} = \Upsilon_{u'}$, but

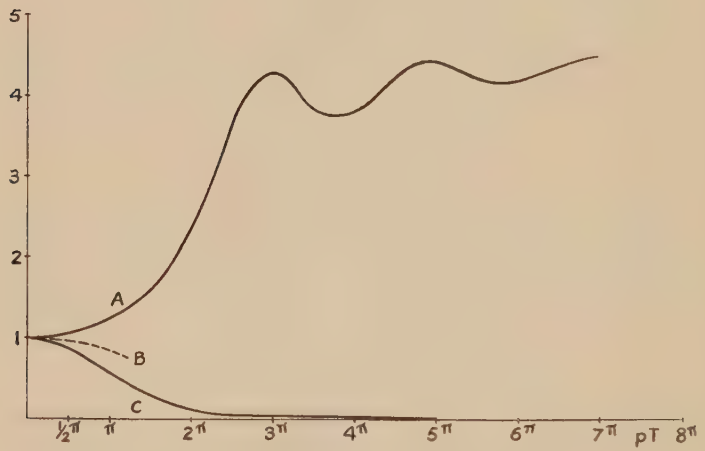
$$\Upsilon_{i'} + \Upsilon_{u'} = 2\Upsilon_\rho^2 \quad (q.\ v.\ \text{Appendix I.}).$$

Fig. 17 indicates that theoretical frequency variations in the space-charge limited case are indeed very different from those in the case of voltage saturation. In addition to the curve for parallel planes (evaluated from the formulæ given in Appendix I. for values of ξ up to 7π) a curve for cylinders (see next section) has been dotted in for small ξ only. As stated above the curve for voltage saturation also applies to the cylindrical case. Curve A of fig. 17 approaches the value 4·5 in a series of oscillations as ξ is increased indefinitely. This would appear to be a point in favour of

the use of equipotential cathode valves as detectors of ultra-short waves.

My experiments on rectification were repeated by Muller and Tank (*loc. cit.*), who also worked with a cylindrical diode. The solution to the cylindrical case is not yet sufficiently advanced to justify an attempt to explain these results. It may, however, be said that the agreement is satisfactory in the case of results obtained for optimum rectification, *i. e.*, at the foot of the anode characteristic. At certain values

Fig. 17.



Theoretical pT variation of the rectification effect.

A. Parallel planes, space-charge limited.

B Cylinders $\left(\frac{R}{a} \rightarrow \infty\right)$, space-charge limited.

C. Planes or cylinders, in the absence of space-charge.

V_0 of anode voltage (in the region of the anode characteristic where the rectification effect passes through zero) the rectification obtained was greater at λ 81 cm. than at $\lambda \infty$ due to a shift in V_0 to some value which was as much as 10 volts higher in the case of λ 81 cm. than in the case of $\lambda \infty$. I am able to confirm that results of this nature actually occur in practice. It is just possible that $\overline{v'}$ builds up to a large value within the system, as fully anticipated in Part I. (p. 657 *et seq.*).

19. An outline of an attempt on the cylindrical case will now be given. The equations corresponding to equations $(1a)$, $(2b)$, $(3a)$ for the plane case are

$$\left.\begin{aligned} \frac{\partial}{\partial r}\left(r\frac{\partial \mathbf{V}}{\partial r}\right) &= -4\pi \mathbf{P}r, \\ \frac{\mathbf{J}}{2\pi r} &= \mathbf{P}\mathbf{U} - \frac{1}{4\pi}\frac{\partial^2 \mathbf{V}}{\partial r \partial t}, \\ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U}\frac{\partial \mathbf{U}}{\partial r} &= -\frac{e}{m}\frac{\partial \mathbf{V}}{\partial r}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (41)$$

in which $\mathbf{J}$, the total current per unit length, is independent of r.

Eliminating $\mathbf{P}$ and $\mathbf{V}$ from the three equations (41), we obtain

$$2\frac{e}{m}\mathbf{J} = \left(\mathbf{U}\frac{\partial}{\partial r} + \frac{\partial}{\partial t}\right)\left(r\mathbf{U}\frac{\partial \mathbf{U}}{\partial r} + r\frac{\partial \mathbf{U}}{\partial t}\right). \quad . \quad . \quad . \quad (42)$$

Proceeding somewhat as in the plane case, we seek the value of $\chi\left(\frac{u_\eta}{u_0}\right)$ in which u_η is the first order velocity corresponding to η. We have used η rather than ξ as the values of T differ in plane and cylindrical cases. The differential equation in $\chi\left(\frac{u_\eta}{u_0}\right)$ is as follows :—

$$\frac{2}{3}\left(\frac{\mathrm{Z}}{r\mathrm{Z}'^2}\right)\eta^2\frac{d^2\chi}{d\eta^2} + \left(1 + \frac{\mathrm{Z}}{r\mathrm{Z}'}\right)\eta\frac{d\chi}{d\eta} + \chi = \cos\eta \sin pt' + \sin\eta\cos pt', \quad . \quad . \quad . \quad . \quad (43)$$

where

$$\left\{\begin{aligned} &\eta = p\mathrm{T}, \\ &t' = t - \mathrm{T} \text{ and is independent of } \eta, \\ &\mathrm{Z} = r\beta^2, \\ &\mathrm{Z}' = \frac{d\mathrm{Z}}{dr}. \end{aligned}\right.$$

I have found it inadvisable to worry about the actual form of β^2, but to concern myself only with the equation which β^2 satisfies. Both have been given by Langmuir, and β^2 may be regarded as thoroughly determined in terms of $\frac{r}{a}$ (a = cathode radius). It is to be noted that β^2 varies across the space between the cylinders, which is a matter of

importance to the solution of (43). Langmuir's equation in β^2 * may be transformed into a rather more simple equation in Z, which reads as follows :—

$$\frac{d^2Z}{dr^2} - \frac{1}{3Z}\left(\frac{dZ}{dr}\right)^2 + \frac{1}{r}\frac{dZ}{dr} - \frac{2}{3r} = 0. \quad . \quad . \quad (44)$$

(43) may be expressed in terms of the single variable r provided we can evaluate T in terms of r. In order to do this we note that (44) may be written

$$Z^{1/3}\frac{d}{dr}(rZ^{-1/3}Z') - \frac{2}{3} = 0,$$

which yields the integral

$$\int_a^r \frac{dr}{Z^{1/3}} = \left|\frac{3}{2}rZ^{-1/3}Z'\right|_a^r = \frac{3}{2}rZ^{-1/3}Z'. \quad . \quad . \quad (45)$$

Now the solution for the steady component of velocity in the cylindrical case may be obtained from the solution given by Langmuir †, and is simply

$$U = \alpha Z^{1/3},$$

where

$$\alpha = \left(9\frac{e}{m}I\right)^{1/3}.$$

Thus

$$T = \int_a^r \frac{dr}{\alpha Z^{1/3}}.$$

The value of $\eta\ (=pT)$ is thus, using (45),

$$\eta = \frac{3p}{2\alpha}rZ^{-1/3}Z'. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

(43) may then be transformed into the following equation :—

$$3rZ\chi'' + (3Z + 4rZ')\chi' + 2\chi = 2\cos\frac{3prZ^{-1/3}Z'}{2\alpha}\sin pt' + 2\sin\frac{3prZ^{-1/3}Z'}{2\alpha}\cos pt',$$

in which all differentiations are now with respect to r. It is evident that we do not gain much by expressing the equation in terms of the single independent variable r. It would be far more satisfactory to have the equation

* Langmuir, Phys. Rev. (2) xxii. p. 348 (1923).
† *Loc. cit.*

entirely in terms of η as the independent variable *. Unfortunately this is not possible. The most hopeful procedure is to treat (43) and (44) as a pair of simultaneous equations. The general solution has not yet been obtained, but one solution which satisfies the equations has been found. Unfortunately I have not arrived at the solution of the equation in the case of r.h.s. $=0$, so that the particular integral obtained is not at present of much value. The particular integral was only obtained at very low values of η such that η^2 and higher powers in the expansion of η are negligible. Its value is as follows :—

$$\chi = \sin pt' + \left(\frac{1}{2} - \frac{1}{3Z'} + \frac{rZ'}{6Z}\right)\eta \cos pt'$$

$$= \sin pt - \frac{1}{2}\left(1 + \frac{2}{3Z'} - \frac{rZ'}{3Z}\right)\eta \cos pt.$$

When we come to investigate the value of the quantity in round brackets we find that at the plane end of the scale $\left(\frac{r}{a} \longrightarrow 1\right)$

$$\frac{1}{2}\left(1 + \frac{2}{3Z'} - \frac{rZ'}{3Z}\right) \longrightarrow \frac{1}{6}\left(2 - \frac{1}{\beta}\right) \longrightarrow \infty.$$

The expression should of course vanish, and the fact that it does not do so indicates that the solution is incomplete. In order to obtain a solution with physical significance we may try writing

$$u_\eta = \alpha_1(\chi Z^{1/3} + A),$$

where A is an arbitrary constant which must make u_η vanish when $r \longrightarrow a$. We then find

$$u_\eta = \alpha_1 Z^{1/3}\left[\sin pt - \eta\left\{\frac{2}{3rZ'} + \frac{1}{2}\left(1 + \frac{2}{3Z'} - \frac{rZ'}{3Z}\right)\right\}\cos pt\right]. \quad . \quad . \quad . \quad . \quad (47)$$

This solution is possibly of some value as far as it goes, since at the plane end the quantity in square brackets reduces to

$$\left[\sin pt - \frac{1}{2}\eta \cos pt\right],$$

which agrees with the plane solution.

When $\frac{r}{a} \longrightarrow \infty$, (47) reduces to

$$u_\eta = \alpha_1 r^{1/3}\left[\sin pt - \frac{2}{3}\eta \cos pt\right], \quad . \quad . \quad . \quad (48)$$

* The independent variable t', present in any case, is excluded from this argument.

which is certainly the correct value for this case (*q. v.*). For values of $\frac{r}{a}$ between $\frac{r}{a}=1$ and $\frac{r}{a}=0$ (47) is, however, not exact. We may obtain a more complete solution for the case $\frac{r}{a}\longrightarrow\infty$ by giving the shape factors in (43) their limiting values at $\frac{r}{a}=\infty$ and treating them as constants during the integration of the equation. As the electron describes its path between the cylinders $r=a$ and $r=\infty$ the shape factors vary between certain limits, as follows :—

$$\frac{1}{6}<\frac{2}{3}\left(\frac{\mathrm{Z}^{\prime}}{r\mathrm{Z}^{\prime 2}}\right)<\frac{2}{3};\quad 1<\left(1+\frac{\mathrm{Z}}{r\mathrm{Z}^{\prime}}\right)<2.$$

In the case $\frac{r}{a}\longrightarrow\infty$ the electron is in a region corresponding to the values $\frac{2}{3}$ and 2 over the major portion of its path. Adopting these values, the equation for $\frac{r}{a}\longrightarrow\infty$ becomes

$$\frac{2}{3}\frac{d^2\chi}{d\eta^2}+2\eta\frac{d\chi}{d\eta}+\chi=\cos\eta\sin pt'+\sin\eta\cos pt,\quad (43\,a)$$

the solution to which is the real part of the expression

$$\chi=\frac{3}{2}\,k\frac{e^{i(pt-\eta)}}{\eta}\left[\eta^{-ik}\int_0^{\eta}\zeta^{ik}\exp(i\zeta)d\zeta-\eta^{ik}\int_0^{\eta}\zeta^{ik}\exp(i\zeta)d\zeta\right],\quad (49)$$

where

$$i=\sqrt{-1},$$
$$k=\sqrt{\tfrac{1}{2}}.$$

The integrals concerned are incomplete gamma functions, and may be evaluated when expressed as the difference to two integrals of the type

$$\int_0^{i\infty}\zeta^{ik}\exp(i\zeta)d\zeta-\int_{\eta}^{i\infty}\zeta^{ik}\exp(i\zeta)d\zeta.$$

The first of these integrals has the value

$$-e^{-\pi^{k/2}}\Gamma(1+ik),$$

while the second yields an asymptotic series applicable only for large values of η.

The complete asymptotic solution is accordingly

$$\chi=\frac{3k}{2i}\frac{e^{i(pt-\eta)}}{\eta}\left[\exp(i\eta)\psi(k,\eta)-\eta^{-ik}e^{-\pi^{k/2}}\Gamma(1+ik)+\eta^{ik}e^{\pi^{k/2}}\Gamma(1-ik)\right],\quad (50)$$

where

$$\psi(k,\eta)-\frac{\alpha_1(k)}{\eta}+\frac{\alpha_2(k)}{\eta^2}+\ldots\ldots+\frac{\alpha_{n-1}(k)}{\eta^{n-1}}\leq \mathrm{R}_n(k),$$

$$i^n\alpha_n(k)=\frac{\Gamma(n-ik)}{\Gamma(-ik)}-\frac{\Gamma(n+ik)}{\Gamma(ik)},$$

$$\mathrm{R}_n(k)\doteqdot\frac{(n-1)!}{|\eta|^n}(1+e^{k\pi/2})\left(\frac{k\sinh k\pi}{\pi}\right)^{1/2}.$$

The best value of the remainder $\mathrm{R}_n(k)$ is given when

$$n=[\eta],$$

where $[\eta]$ denotes the integral part of η.

Inspection of (50) reveals that $\eta\chi$ represents an oscillation about the axis of η. The amplitude of $\eta\chi$ is sensibly constant, but the period increases with η.

In order to apply the solution (50) it would be necessary (in the absence of tables of Γ-functions of an imaginary argument) to evaluate $\Gamma(n+ik)$ for some large value of n, for which the formula

$$\log_e\Gamma(x+1)$$
$$=\log_e\sqrt{2\pi}+\left(x+\frac{1}{2}\right)\log_e x-x+\frac{\mathrm{B}_1}{1.2.x}-\frac{\mathrm{B}_2}{3.4.x^3}+\frac{\mathrm{B}_3}{5.6.x^5},$$

where B's are Bernoulli's numbers, being coefficients in the expansion of

$$\frac{2}{x}\frac{e^x+1}{e^x-1}$$

in the form

$$1+\mathrm{B}_1\frac{x^2}{2!}-\mathrm{B}_2\frac{x^4}{4!}+\ldots\ldots,$$

gives results to the required degree of accuracy, and then to proceed by applying the recurrence formula

$$\Gamma(1+z)=z\Gamma(z)$$

until the value $n=1$ is reached. With each application of the recurrence formula a considerable amount of rationalization is involved.

For present purposes, therefore, I have deemed it not worth while to attempt to evaluate the functions appearing in the asymptotic solution, but to content myself with a simple power series solution of (49), which is applicable for small values of η, *e.g.*,

$$\chi = S_u \sin pt + S_u' \cos pt,$$

where

$$\left.\begin{aligned} S_u &= \left(1 - \frac{14}{57}\eta^2 + \frac{412}{31977}\eta^4 - \ldots\ldots\right), \\ S_u' &= -\frac{2}{3}\left(\eta - \frac{20}{209}\eta^3 + \ldots\ldots\right). \end{aligned}\right\} \quad . \quad . \quad . \quad (51)$$

When arranged for optimum convergence the power series solution (51) can be made to yield accurate results over the range $0 < \eta < 5$, a range which is sufficiently representative for the purpose of comparison with the solution for planes. (51) can be arranged for optimum convergence by expressing the solution for χ in the form

$$\chi = S \sin (pt + q\eta) + S' \cos (pt + q\eta).$$

The value of q for optimum convergence is in this case $\frac{2}{3}$, *i.e.*, is equal to the coefficient of η in the coefficient S_u' of $\cos pt$. The series S, S′ are as follows :—

$$\left.\begin{aligned} S &= 1 - \frac{4}{171}\eta^2 + \frac{210}{863379}\eta^4 - \ldots\ldots, \\ S' &= -\frac{20}{16929}\eta^2 + \ldots\ldots \end{aligned}\right\} \quad . \quad . \quad . \quad (52)$$

The convergence of S, S′ is very much better than that of S_u, S_u', which are therefore evaluated from the relations

$$\left.\begin{aligned} S_u &= S \cos q\eta + S' \sin q\eta, \\ S_u' &= S' \cos q\eta - S \sin q\eta, \end{aligned}\right\} \quad . \quad . \quad . \quad (53)$$

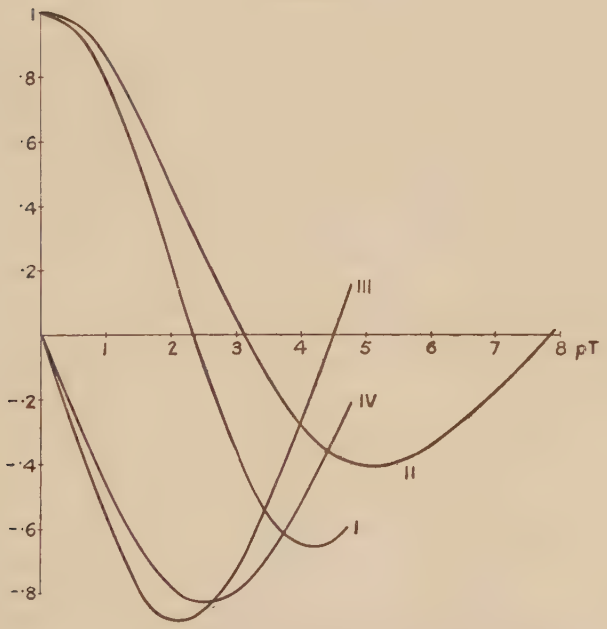
where $q = \frac{2}{3}$.

Fig. 18 compares the series S_u, S_u' in the present case with the corresponding series in the case of parallel planes. The similarity in form is perhaps more striking than would have been anticipated from comparison of the solutions for plane and cylindrical structures. This similarity is even better demonstrated by comparison of S_u, S_u' (cylinders) with S_i, S_i' (planes) (*cf.* fig. 8), and suggests that the formulæ for S_i, S_i' (planes) might be used to extend the range of the solution in the cylindrical case. While there is no mathematical justification for the extension of an approximation formula in this manner, we should for physical reasons

predict a similarity in form between the functions on the grounds that, in each of the two cases compared, it is the time of transit and not the shape of the structure which occurs explicitly.

An examination of the second-order solution leads eventually to a result for the rectification effect which shows that in the case of cylinders for which $\frac{R}{a} \longrightarrow \infty$ a decrease with

Fig. 18.



Curves I., II. represent S_u for cylinders, planes respectively.
Curves III., IV. represent S_u' for cylinders, planes respectively.

frequency takes place ; for small values of η only the result obtained is

$$\Upsilon_{\Delta 1} = 1 - \frac{35\eta^2}{684} \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Further examination of the solution in the case for which $\frac{r}{a}$ has finite values leads to a formula for the dielectric constant which is believed to be very nearly exact :

$$\left.\begin{aligned} \epsilon &= \bar{C}/C_1, \\ \bar{C} &= \frac{F(\beta)}{\beta}\left\{\frac{3}{8} - \frac{1}{4}\left(\frac{1}{F} - \frac{F}{4}\right)\right\}, \end{aligned}\right\} \quad \text{where} \quad \left.\begin{aligned} F(\beta) &= \beta + 2r\frac{d\beta}{dr}, \\ C_1 &= \frac{1}{2\log\frac{r}{a}}. \end{aligned}\right\} \quad . \quad . \quad . \quad (55)$$

The above formula gives $\epsilon \frac{3}{5}$ at $\frac{r}{a}=1$. At $\frac{r}{a}=\infty$ the "hot" capacity remains finite while the "cold" capacity

TABLE VII.

$\frac{r}{a}$.	$\epsilon(=\bar{C}/C_1)$.	$\frac{C}{C_1}$.
1	0·6	1·3
5	0·64	1·60
10	0·67	1·85
20	0·71	2·12
50	0·77	2·61
100	0·88	2·98
200	0·98	3·43
230	1·00	3·50
1000	1·27	4·54
10000	1·72	6·13
∞	$\log\infty$	$\log\infty$

approaches zero logarithmically. The dielectric constant thus approaches infinity logarithmically as $\frac{r}{a} \longrightarrow \infty$, a fact which is at first sight somewhat surprising. A little consideration shows that it is the distribution of electrons throughout the space which is responsible for a non zero value of "hot" capacity under all conditions.

Table VII. reveals that in most practical cases

$$\left(1 < \frac{r}{a} < 50\right),$$

the dielectric constant is not very different from ·6. The

last column gives "true" capacity ÷ "cold" capacity, from the (exact) formula

$$\frac{C}{C_1} = \frac{2}{3}\frac{F(\beta)}{\beta}\log\frac{r}{a},$$

and is the one to go by if the capacity between the negative grid of a triode and some other electrode is contemplated. These values indicate that the grid cathode capacity in a practical case may be as much as three times the "cold" capacity. Experiments showing that this is so have already been described (section 16). With regard to the structure employed, it should be emphasized that the form of the *cathode* is the primary consideration, so that filamented valves may be regarded as conforming to a cylindrical structure more nearly than a plane structure, even though the anode be plane.

In conclusion I should like to express my indebtedness to Mr. D. P. Dalzell for quoting the asymptotic solution (50), and to Messrs. Merleau and Bohnenblust for performing the somewhat laborious calculations leading to the middle column of Table VII. I should like to thank Mr. B. S. Gossling, M.A., and Mr. L. H. Bedford, M.A., for valuable advice and criticism.

5th June, 1930.

Appendix I.

(*a*) Υ-functions.

The functions are given in order of importance in respect of variation with ξ.

$$\Upsilon_p^2 = \frac{36}{\xi^6}\left[\frac{\xi^4}{4} + \xi^2\cos\xi - 2\xi\sin\xi + 2(1-\cos\xi)\right],$$

$$\Upsilon_v^2 = \frac{4}{\xi^8}[\xi^6 + 12\xi^4(1+\cos\xi) - 24\xi^3\sin\xi + 72\xi^2(1+\cos\xi) - 288\xi\sin\xi + 288(1-\cos\xi)],$$

$$\Upsilon_u^2 = \frac{72}{\xi^6}[\xi^2(1+\cos\xi) - 4\xi\sin\xi + 4(1\cos\xi)],$$

$$\Upsilon_i^2 = \frac{4}{\xi^4}[\xi^2 - 2\xi\sin\xi + 2(1-\cos\xi)].$$

The Υ-functions of the second order solution are expressible in terms of the first order Υ-functions, as follows :—

$$\Upsilon_u' = \Upsilon_u^2,$$

$$\Upsilon_i' = 2\Upsilon_\rho^2 - \Upsilon_u^2,$$

$$= \frac{18}{\xi^2}(1-\Upsilon_i^2),$$

$$\Upsilon_{\Delta I} = \frac{\Upsilon_i'}{\Upsilon_v^2}.$$

Of the two alternative expressions for Υ_i' the first is less easy to interpret. We may also write this :

$$\tfrac{1}{2}(\Upsilon_i' + \Upsilon_u') = \Upsilon_\rho^2.$$

We may regard the second expression for Υ_i' as stating that $\xi^2\Upsilon_i'$ is proportional to the difference between the squares of the moduli of total current and of conduction current, remembering, however, that $1-\Upsilon_i^2$ does *not* give the displacement current (*cf.* equation (9)). It may be inferred, however, that the ξ variation of the rectification effect is in some way connected with that of the dielectric constant, as by reference to figs. 5*a* and 17 it will be noticed that maxima and minima occur at similar values of ξ.

(*b*) Ξ-functions.

$$\Xi_r = \tan^{-1}\frac{S_r'}{S_r},$$

in order of importance of ξ variation.

r.	S_r.	S_r'.
v	$\frac{12}{\xi^4}(2-2\cos\xi-\xi\sin\xi)$	$\frac{12}{\xi^4}(2\sin\xi-\xi-\xi\cos\xi)-\frac{2}{\xi}$
u	$\frac{6}{\xi^2}(-\cos\xi+\frac{2}{\xi}\sin\xi-1)$	$\frac{6}{\xi^2}\left(\sin\xi+\frac{2}{\xi}\cos\xi-\frac{2}{\xi}\right)$
i	$\frac{2}{\xi^3}(\xi\sin\xi-1+\cos\xi)$	$\frac{2}{\xi^2}(\xi\cos\xi-\sin\xi)$
ρ	$\frac{3}{\xi^3}(2\xi\cos\xi-2\sin\xi+\xi^2\sin\xi)$	$\frac{-3}{\xi^3}(2\xi\sin\xi-\xi^2\cos\xi-2+2\cos\xi)$

Appendix II.

The formula

$$I = \frac{\sqrt{2}}{9\pi} \cdot \sqrt{\frac{e}{m}} \cdot \frac{V^{3/2}}{d^2}$$

may also be written

$$I = \pi . \sqrt{2} . \sqrt{\frac{e}{m}} . C^2 . V^{3/2},$$

where

$$C = \frac{4}{3} \times \text{cold capacity.}$$

In extending the space-charge theory to triodes the assumption is frequently made * that the combined effect of anode and grid can be represented by writing

$$V_s = \frac{V_b}{\mu_1} + V_c,$$

where V_b, V_c are anode and grid potentials, μ_1 is the voltage amplification factor. Under these conditions V_s is the "lumped" voltage at the grid. Writing $C_{eff.}$ for the effective input capacity of the triode, the following is an approximation to the anode current:—

$$I_p = A C_{eff.}^2 V_s^{3/2},$$

where A is some constant whose value need not concern us.

We now make small changes in V_b and V_c such that $dI_p = 0$. Thus

$$0 = C_{eff.}^2 \cdot \frac{3}{2} V_s^{1/2} \left(\frac{dV_b}{\mu_1} + dV_c \right) + 2 C_{eff.} V_s^{3/2} dC_{eff.},$$

which yields the result

$$\frac{-dV_b}{dV_c} = \mu_1 \left(1 + \frac{4}{3} \frac{V_s}{C_{eff.}} \frac{dC_{eff.}}{dV_c} \right),$$

in which μ_1 is the amplification factor at "cut-off" ($V_s = 0$), $\frac{dC_{eff.}}{dV_c}$ is the rate of change of input capacity with grid voltage.

The variation is in the right direction, since dV_c is positive

* *Cf.* Eccles, Rad. Rev. 1919.

when the negative grid bias is decreased. On referring to fig. 15 we note from the upper curve that at $V_c=0$

$$\frac{d}{dV_c}\frac{C}{C_1} \doteqdot 0{\cdot}1 \text{ volt}^{-1}.$$

Taking

$$\left.\begin{aligned} V_s &= 20, \\ \frac{C_1}{C} &= \frac{1}{2{\cdot}14}, \end{aligned}\right\},$$

$$\frac{4}{3}\frac{V_s}{C}\frac{dC}{dV_c} = 1{\cdot}24,$$

so that

$$\mu = -\frac{dV_b}{dV_c} = 2{\cdot}24\,\mu_1.$$

For values of V_c between $V_c=0$ and cut-off μ has values intermediate between $2{\cdot}24\ \mu_1$ and μ_1. While a μ variation of this magnitude has not been observed experimentally with a D.E.P. 215 valve, the following table is of interest. A bridge method was employed, the frequency being 800 cycles. This method was not applicable in the case of anode currents below about 200 μA.

μ Variations of D.E.P. 215 Valve.

$V_a=2$, $V_b=120$, V_c varied.

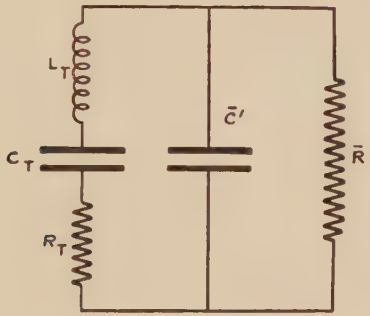
$-V_c$.	I_b. (mA).	μ.
0	19·3	7·18
2	17.2	7·12
4	14·3	7·08
6	11·7	7·02
8	9·1	6·90
10	6·7	6·80
12	4·5	6·69
14	2·7	6·50
16	1·3	6·33
18	0·5	5·95
20	0·2	5·2

There is evidence that the results quoted are not due to end effects. Careful d.c. measurements between $-V_c 20$ and $-V_c 45$ (where I_b is still $32\,\mu A$) reveal that μ approaches the value 2 at $V_c - 35$ and remains constant as $-V_c$ is further increased. This last result, together with the existence of a residual anode current below the true cut-off, must be interpreted as due to end-effect.

(N.B.—The above μ variations are quite representative of filamented valves, and are by no means peculiar to the particular type chosen. In general the μ variations are more pronounced the smaller the grid-cathode clearance.)

It must be pointed out that the problem of the variation of μ is exceedingly complicated in general owing to the finite size of the grid wires.

Fig. 19.



Appendix III.

The representation of the external circuit constituted by a Lecher wire system is not straightforward in view of distributed inductance and stray capacities.

The circuit of fig. 19, in which a condenser C_T is placed in series with L_T, R_T, is somewhat more general than that of fig. 16, section 17. The introduction of the series condenser C_T raises the order of the system from 2 to 3 *, the operational equation being

$$D^3 + \left(\frac{R_T}{L_T} + \frac{1}{C'\bar{R}}\right) D^2 + \left(\frac{1 + \frac{\bar{C}'}{C_T} + \frac{R_T}{\bar{R}}}{L_T C'}\right) D + \frac{1}{L_T C_T \bar{C}'\bar{R}} = 0.$$

* *Cf.* Tellegen, *Archiv. f. Electrotechnik*, xxii. p. 1 (1929).

If we place $C_T=\infty$ the equation reduces to

$$D^2+\left(\frac{R_T}{L_T}+\frac{1}{\overline{C'}\overline{R}}\right)D+\frac{\left(1+\frac{R_T}{\overline{R}}\right)}{L_TC'}=0.$$

In order that oscillations may be sustained the coefficient of D must vanish. We thus arrive at the condition (33) of section 16. The equation then reduces to

$$D^2-k^2=0,$$

where

$$k=\sqrt{-\frac{1}{L_T\overline{C'}}\left(1+\frac{R}{\overline{R}}\right)},$$

the solution to which is of the form $Ae^{\pm Kt}$.

If K is imaginary, undamped oscillations of frequency p occur, where $jp=k$ ((32), section 17).

It will be noticed that the coefficient of D^2 in the cubic equation is the same as that of D in the quadratic. We may therefore omit the term concerned, and the cubic reduces to standard form, *e. g.**,

$$y^3-3qy-2r=0,$$

in our case

$$q=\frac{-\left(1+\frac{\overline{C'}}{C_T}+\frac{R_T}{\overline{R}}\right)}{3L_T\overline{C'}},$$

$$r=\frac{-1}{2L_TC_T\overline{C'}\overline{R}}.$$

In view of the additional positive term $\frac{\overline{C'}}{C_T}$ in the expression for q, the condition that the negative value of $\overline{R}$ should not be such as to make the expression in round brackets negative can be assumed satisfied in this case, as in the case $C_T=\infty$ already considered. Thus q will be negative. Writing †

$$\sinh\phi=\frac{r}{\sqrt{(-q^3)}},$$

the roots of the equation are

$$y_1=2\sqrt{(-q)}.\sinh\left(\tfrac{1}{3}\phi\right),$$

* *Cf.* 'Smithsonian Mathematical Formulæ,' p. 9.

† 'Smithsonian Mathematical Formulæ,' p. 10.

$$y_2 = -\frac{y_1}{2} + i\sqrt{(-3q)}.\cosh(\tfrac{1}{3}\phi),$$

$$y_3 = -\frac{y_1}{2} - i\sqrt{-3q}.\cosh(\tfrac{1}{3}\phi),$$

where all quantities are real except $i\,(=\sqrt{-1})$. The real root y_1 corresponds to a "building up" term of the type Ae^{kt}, while the complex roots y_2 and y_3 combine to give an oscillation which is more or less damped according to the value of y_1. When $r=0$ we are left with two equal and opposite roots, the same as in the quadratic case. In this case we get an undamped oscillation, but in general undamped oscillations cannot occur. Evidently the coefficient of D^2 in the cubic equation would have actually to become negative in order to overcome the damping arising from the introduction of the series capacity C_T.

Appendix IV.

Consider a thermionic system, hereafter referred to simply as a *system*, consisting of a pair of *circular* parallel plates. Imagine a large number of such systems placed end to end so as to form a long conductor of circular section. Regarding each system as an element of length l_0 of an infinitely long conductor without return conductor, and neglecting edge effects of the plates, which are supposed of radius l_r where $l_r \gg l_0$, we have for the radiation resistance r_j of a single system (Steinmetz, 'Transient Electric Phenomena and Oscillations,' 3rd ed., p. 400)

$$rj = 2pl_0 \operatorname{col}\frac{pl_r}{c} \times 10^{-9} \text{ ohm}, \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{where col } x = \frac{\pi}{2} - \int_0^x \frac{\sin u}{u}\,du.$$

In a footnote to equation (ii.) we defined the quantity, R_0S_v as the "effective resistance to the current j." If each system is in a state of maximum negative R_0S_v we find from Table III.

$$pT = 7{\cdot}2\ ;\ -R_0S_v = {\cdot}022R_0 = -R_j \text{ (say)}.$$

Taking, as in section 17, $l_0 = {\cdot}03$ cm., and using (18)

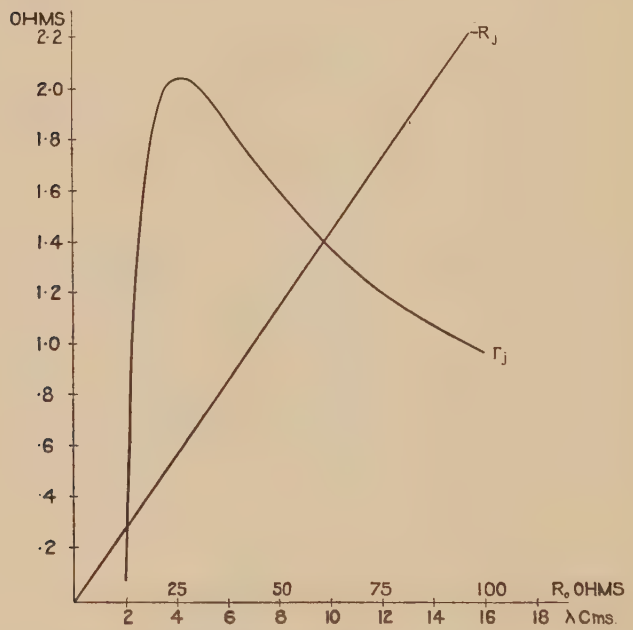
$$R_0 = 50\frac{\lambda}{7{\cdot}72},$$

so that

$$-R_j = 1{\cdot}1\left(\frac{\lambda}{7{\cdot}72}\right). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now (2) applies to plates of unit area (neglecting edge effects) so that (1) will apply when $l_r = \frac{1}{\sqrt{\pi}}$ cm. Fig. 20, on which r_j and $-R_j$ are plotted as functions of λ, indicates that the radiation resistance of a system is a by no means negligible quantity, but may actually exceed the (maximum attainable) negative resistance over a certain range of wavelengths. Above λ 9·8 cm., where the curves cross r_j decreases continuously, the decrease becoming hyperbolic

Fig. 20.



for λ large. Fig. 20 is, of course, liable to large corrections in practical cases where we are concerned with a single system associated with finite (go and return) conductors, and is only intended to indicate the order of events. In case $r_j \geq -R_j$ oscillations cannot occur.

In conclusion, it is worth while to satisfy ourselves that the equivalent radiation resistance of an accelerated electron can be neglected. The total energy radiated in unit time is given by Larmor's formula

$$\frac{2}{3} \cdot \frac{e^2}{c^3} \cdot \dot{U}^2.$$

In the case of uniform acceleration from rest to a final velocity U,

$$\dot{U} = \frac{U}{T} = \text{constant.}$$

The current I due to the electron being $\frac{2e}{T}$ we find, for the "radiation resistance" in ohms,

$$\frac{U^2}{1{\cdot}8 \times 10^{20}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For $U = 10^9$ cm. per sec. we find that the "radiation resistance" is only $5{\cdot}5 \times 10^{-3}$ ohm, *e. g.*, is of negligible importance.

XL. *A Source of Mechanical Vibration for Experimental Purposes.* *By* F. Aughtie, *Ph.D., M.Sc.* (*The National Physical Laboratory, Engineering Dept.*) *.

Abstract.

Following a brief *résumé* of early attempts to vibrate a loaded beam in a vertical plane, which were unsuccessful, due either to bad wave form or the presence of excessive horizontal movement, a description is given of the final satisfactory method, which gave a controllable amplitude up to 0·002 inch at frequencies from 8 to 35 cycles/sec. with negligible horizontal movement and good wave form. The necessary force for vibrating the beam was obtained from resonant vibrations of an auxiliary mass-spring system tuned to the working frequency. The oscillations were maintained by a small crank and electric motor, and an important feature was the use of solid friction to give a true flat-topped resonance curve, thus permitting small changes of motor speed without variation of amplitude. The equivalent electrical circuit is given of the mechanical filter system used, and records are reproduced of wave forms obtained with different methods.

In connexion with some recent vibration experiments it was necessary to vibrate a loaded beam vertically through an amplitude of a few thousandths of an inch

* Communicated by the Author.

at frequencies from 8 to 35 cycles/sec. It was further necessary for the movement to be closely sinusoidal with time and be unaccompanied by any appreciable horizontal movement. As this problem proved far more difficult of solution than appeared at first sight, the following note on some methods tried and a description of the final arrangement may be of interest.

An out-of-balance weight attached to the shaft of a small motor clamped to the beam was first tried as a source of vibration, but, while giving a good wave form at most frequencies, had to be abandoned on account of the excessive horizontal movement, which was not sufficiently eliminated by suspending the motor from a flexible strut. To avoid this horizontal movement the crank was then used to extend a spring whose other end was attached to the beam, but it was not found possible to obtain a sufficiently large force in this way without overloading the motor bearings. (In order to keep the connecting rod always in tension, a necessary condition to eliminate backlash, the maximum load on the bearings is more than twice that due to an out-of-balance weight giving the same vibrating force.)

A further attempt to eliminate horizontal movement made use of a mass suspended from the crank by a long connecting rod, the motor being mounted on the beam as in the first method. At crank speeds above that corresponding to the time of oscillation of the pendulum formed from the rod and mass the latter had negligible horizontal movement, and the force necessary to accelerate it was therefore almost entirely in a vertical direction. This method had to be abandoned on account of the bad wave form, due most probably to the effect of slight backlash in the bearings.

The arrangement illustrated in fig. 1 was then tried, and gave satisfactory results. The force required for vibrating the beam A was provided by the deflexion of a cantilever spring C, but instead of deflecting it with a crank it was loaded with a mass D, so chosen to give a natural frequency equal to that required for the experiment. The operating frequency was readily changed by varying the length of the spring.

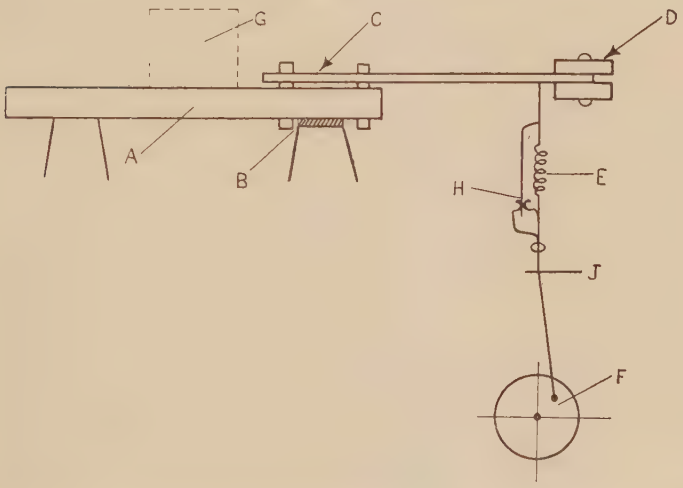
A rubber pad B was inserted at one pier to permit sufficient amplitude of the beam at low frequencies.

Vibrations of the system C–D were initiated through

a forcing spring E operated by a variable-throw crank on the shaft of a variable speed electric motor, a guide being added at J to prevent side movement of the forcing spring.

If the damping of the system C–D is small, at a motor speed corresponding to the natural frequency f_0 of this system, the amplitude of the oscillations will be equal to the throw of the crank. At some speed slightly above this the mechanical impedance of the system is equal to that of a small mass, and this will have a natural frequency f_1 with the stiffness of the forcing spring. At a motor speed corresponding to this frequency f_1 a secondary

Fig. 1.



resonance will occur, and the oscillations will attain an amplitude usually greater than the throw of the crank, the actual value depending only upon the damping of the system.

The conditions above described are presented in fig. 2, where in the upper half are plotted, (*a*) the force necessary to oscillate the mass through unit distance, and (*b*) that required to give unit deflexion of the spring. The intersection of *a* and *b* gives the true resonant frequency f_0 of the system. Above *b*, and separated from it by a distance corresponding to the stiffness of the forcing spring, is a third line, *c*, whose intersection with *a* gives the apparent resonant frequency f_1 of the system, *i.e.*, the

frequency at which the amplitude of the oscillations is a maximum.

The lower half of the figure illustrates the relation between amplitude and frequency, and it is clear that

Fig. 2.

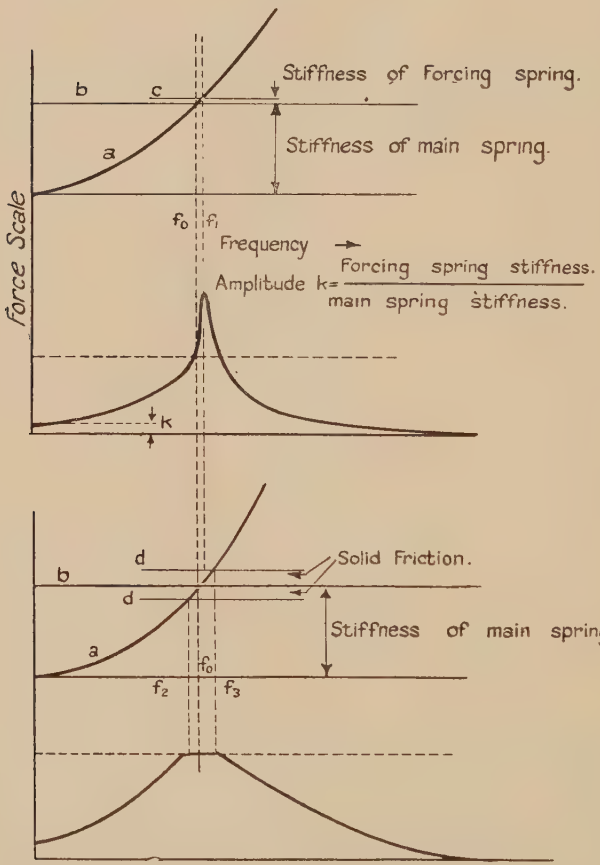


Fig. 3.

unless very close control of motor speed is possible the system is useless as a source of vibration of constant amplitude.

To overcome this defect some solid friction was added at H (fig. 1) across the ends of the forcing spring. The effect of this addition is shown in fig. 3. In the upper half

of this figure the two curves a and b are repeated, and two lines, d d, are added displaced above and below b by a distance corresponding to the magnitude of the solid friction. These two lines intersect a in two frequencies, f_2 and f_3, between which the force necessary to maintain forced oscillations of the system is less than the solid friction. Between these two frequencies there is therefore no extension of the forcing spring, and the amplitude of the oscillations is equal to the throw of the crank. The lower half of the figure shows, as before, the relation between amplitude and frequency. In practice, owing to the finite stiffness of the connecting rod, the top of the curve is slightly sloping instead of truly horizontal, but the departure can be made negligibly small.

Fig. 4.

Fig. 5.

Fig. 6.

Frequency 24 cycles/sec.

The maximum load on the motor bearing occurs in practice at low speeds and is the sum of the solid friction and the force necessary to extend the forcing spring. This sum can readily be made of the order of one tenth of the total force available for vibrating the apparatus.

By using a connecting rod long compared with the throw of the crank, and keeping f_2 and f_3 as close together as is practicably consistent with unavoidable changes of motor speed, distortion of the wave form can be kept extremely small. Typical wave forms (obtained with a vibration-recording instrument) are reproduced in figs. 6–8. Examples of bad wave forms produced by other methods are illustrated in figs. 4 and 5.

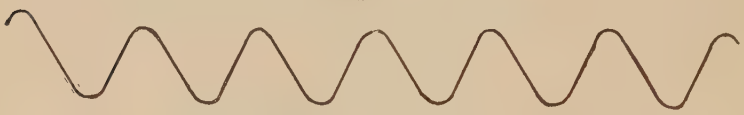
As a matter of interest the equivalent electrical circuit is shown in fig. 9, where G is a generator giving constant displacement of charge, *i. e.*, a current increasing as the frequency.

C_1 is a (small) condenser corresponding to the stiffness of the cantilever.

L is an inductance corresponding to the equivalent mass of the mechanical system.

C_2 is a (large) condenser corresponding to the stiffness of the forcing spring.

Fig. 7.

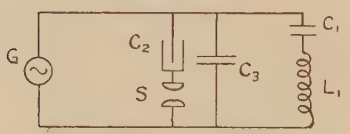


Frequency 33·75 cycles/sec.

Fig. 8.

Frequency 8·2 cycles/sec.

Fig. 9.



S is a spark gap corresponding to the solid friction.

C_3 is a (very small) condenser corresponding to the finite stiffness of the connecting rod.

Summarizing, the method described gives an oscillating force of good wave form whose magnitude is of the order of ten times the load on the motor bearings and which remains practically constant over a small range of motor speed such as may arise from voltage fluctuations.

October 15, 1930.

XLI. *The Application of Functional Operations to a Class of Integral Equations occurring in Physics**. *By* Henry P. Thielman, *M.A.*†

Introduction.

THE object of this paper is to apply generalized differentiation and integration to a class of integral equations occurring in physics. Before, however, we go over to the main topic of our investigation we shall note certain relations existing between Volterra's functional composition and the operations of generalized differentiation and integration as originally defined by Riemann ‡, and modified by H. T. Davis §, Paul Levy‖, and others.

It must be said that some of the integral equations considered here have been studied by D. M. Wrinch and J. W. Nicholson in a recent paper which appeared in the 'Philosophical Magazine'¶. The methods employed there are, however, quite laborious, and the main results obtained lack a certain amount of generality which can be obtained by the methods used here. The application of functional operations to the class of integral equations under consideration simplifies the work of deriving the main results stated in the above-mentioned paper so considerably that a reconsideration of the whole topic seems justifiable. We shall, however, consider integral equations of a more general type than those considered in the paper referred to; on the other hand, we shall omit the study of applications of the main theorems to functions which are zero for parts of their ranges, since this has been done by Drs. Wrinch and Nicholson in a very complete manner.

* The author is indebted to Professor A. D. Michal for helpful suggestions and valuable criticisms.

† Communicated by the Author.

‡ B. Riemann, "Versuch einer allgemeinen Auffassung der Integration und Differentiation," 'Gesammelte mathematische Werke,' Leipzig, pp. 353–366 (1892).

§ H. T. Davis, "Fractional Operations as applied to a class of Volterra Integral Equations," Amer. Journ. of Math. xlvi. pp. 95–105 (1924).

‖ Paul Lévy, "Sur la dérivation et l'intégration généralisées," *Bulletin des Sciences Mathématiques*, tome xlvii. série 2, p. 307 (1923); "Le calcul symbolique d'Heaviside," *Bulletin des Sciences Mathématiques*, tome l. série 2, p. 174 (1926).

¶ D. M. Wrinch and J. W. Nicholson, "A Class of Integral Equations occurring in Physics," Phil. Mag. iv. (7) pp. 531–560 (1927).

1. *Functional Operations.*

Making use of H. T. Davis's generalization of differentiation and integration, we have the following definitions :—

By *generalized integration* of a function $f(y)$ † with respect to y we shall mean

$$ {}_{x}\mathrm{D}_{y}^{-\nu} f(y) = \int_{x}^{y} \frac{(y-t)^{\nu-1}}{\Gamma(\nu)} f(t)dt, \qquad (\nu>0), \quad (1.1) $$

where x is any constant such that $-\infty \leq x \leq +\infty$.

By *generalized differentiation* of a function $f(y)$ with respect to y we mean

$$ {}_{x}\mathrm{D}_{y}^{m+\nu} f(y) = \frac{d^{m+1}}{dy^{m+1}} \int_{x}^{y} \frac{(y-t)^{-\nu}}{\Gamma(1-\nu)} f(t)\,dt, $$
$$ (0<\nu<1;\ m=0,\ 1,\ 2,\ \ldots;\ -\infty \leq x \leq +\infty). $$
$$ \ldots \quad (1.2) $$

The operation of composition of two functions $f(x, y)$ and $g(x, y)$ is defined by Vito Volterra ‡ as

$$ \int_{x}^{y} f(x, t)\, g(t, y)\, dt, \quad \ldots \quad (1.3) $$

and is denoted by $\overset{*}{f}(x, y)\, \overset{*}{g}(x, y)$ or simply by $\overset{*}{f}\,\overset{*}{g}$. If $g(x, y) = f(x, y)$, the above expression is designated by $\overset{*2}{f}(x, y)$ and is called the second power of composition of $f(x, y)$. The first power of composition of a function $f(x, y)$ is defined as the function itself, *i.e.*, $\overset{*1}{f}(x, y) = f(x, y)$. Additionally we shall use the operator $\overset{*0}{f}(x, y)$, which is called the zeroth power of composition of $f(x, y)$, and is defined by the relations

$$ \overset{*0}{f}(x, y)\, \overset{*}{g}(x, y) = \overset{*}{g}(x, y)\, \overset{*0}{f}(x, y) = g(x, y) \quad . \quad (1.4) $$

whatever the nature of the functions $f(x, y)$ and $g(x, y)$ may be. We see that the operator $\overset{*}{f}{}^{0}(x, y)$ plays the part of unity in this theory of composition. For this reason it is designated by $\overset{*}{1}{}^{0}$

† All functions occurring in this paper will be assumed to be Riemann integrable unless otherwise specified.

‡ Vito Volterra et Joseph Pérès, 'Leçons sur la composition et les fonctions permutable.'

A function $f(x, y)$ of order α is defined as

$$f(x, y) = \frac{(y-x)^{\alpha-1}}{\Gamma(\alpha)}\phi(x, y), \quad . \quad . \quad . \quad (1.5)$$

where α is not a negative integer or zero, and $\phi(x, x) = 1$. If $\phi(x, y) \equiv 1$, the function $f(x, y)$ becomes

$$f(x, y) = \frac{(y-x)^{\alpha-1}}{\Gamma(\alpha)}, \quad . \quad . \quad . \quad . \quad (1.6)$$

and is denoted by $\overset{*\alpha}{1}$.

Composing $\overset{*\alpha}{1}$ with $\overset{*\beta}{1}$, we have by (1.3)

$$\overset{*\alpha}{1}\,\overset{*\beta}{1} = \int_x^y \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\,\frac{(y-t)^{\beta-1}}{\Gamma(\beta)}\,dt,$$

which for $\alpha > 0$, $\beta > 0$ is easily seen to be

$$\frac{(y-x)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}.$$

Hence we have the law:

$$\overset{*\alpha}{1}\,\overset{*\beta}{1} = \overset{*\alpha+\beta}{1}, \qquad (\alpha>0, \beta>0). \quad . \quad (1.7)$$

With Volterra† we define a function $\overset{*-n}{1}$, n a positive integer, so as to satisfy the following laws:—

(1) The formula (1.7) subsists for α and β equal to negative integers.

(2) If H is any function of order α different from a negative integer, and $-n$ is a negative integer, then

$$\left.\begin{aligned} \overset{*-n}{1}\overset{*}{\mathrm{H}} &= (-1)\frac{\partial^n}{\partial x^n}\mathrm{H}(x, y), \\ \overset{*}{\mathrm{H}}\,\overset{*n}{1} &= \frac{\partial^n}{\partial y^n}\mathrm{H}(x, y). \end{aligned}\right\} \quad . \quad . \quad . \quad (1.8)$$

Composing $f(y)$ with $\overset{*\nu}{1}$, $\nu > 0$ we have

$$\overset{*}{f}(y)\overset{*\nu}{1} = \int_x^y \frac{(y-t)^{\nu-1}}{\Gamma(\nu)} f(t)\,dt, \qquad (\nu>0), \quad . \quad (1.9)$$

† Vito Volterra et Joseph Pérès, *loc. cit.* p. 101.

which is equation (1.1). Composing $f(y)$ with $\overset{*}{1}^{-m-\nu}$ $(m = 0, 1, 2, \ldots ; 0<\nu<1)$, we have

$$\overset{*}{f}(y)\overset{*}{1}^{-m-\nu} = \overset{*}{f}(y)\overset{*}{1}^{-(m+1)+(1-\nu)}$$

$$= \frac{d^{m+1}}{dy^{m+1}} \int_x^y \frac{(y-t)^{-\nu}}{\Gamma(1-\nu)} f(t)dt, \quad . \quad . \quad . \quad (1.10)$$

which is equation (1.2). Hence we can state the following :—

THEOREM I.

The operations of generalized differentiation and integration on the one hand, and Volterra's functional composition on the other hand, are connected by the following relations :

$${}_x\mathrm{D}_y^{-\nu} f(y) \equiv \overset{*}{f}(y)\overset{*}{1}^{\nu}, \qquad (0<\nu), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.11)$$

$${}_x\mathrm{D}_y^{m+\nu} f(y) \equiv \overset{*}{f}(y)\overset{*}{1}^{-(m+\nu)}, \qquad (0<\nu<1 ;\ m=1, 2, \ldots).$$

$$. \quad . \quad . \quad (1.12)$$

Having established the identities (1.11) and 1.12), we conclude that all the laws for generalized differentiation and integration are identical with certain laws on functional composition. The latter, however, have been well established and are widely known.

We shall next state some of the properties of our functional operators. For convenience in reference to other papers on the same subject we shall change our notation somewhat, so as to conform to that of other writers. We will set in equation (1.1) and (1.2) $x = c$, $y = x$. Then those equations become

$${}_c\mathrm{D}_x^{-\nu} f(x) = \int_c^x \frac{(x-t)^{\nu-1}}{\Gamma(\nu)} f(t)dt, \quad (0<\nu ;\ -\infty \leq c \leq +\infty),$$

$$. \quad . \quad (1.13)$$

$${}_c\mathrm{D}_x^{m+\nu} f(x) = \frac{d^{m+1}}{dx^{m+1}} \int_c^x \frac{(x-t)^{-\nu}}{\Gamma(1-\nu)} f(t)dt,$$

$$(0<\nu<1 ;\ m=0, 1, 2, \ldots ;\ -\infty \leq c \leq +\infty)$$

$$. \quad . \quad (1.14)$$

As a direct consequence of our definitions we have† the following *laws* :

$${}_c\mathrm{D}_x^{\mu}[u(x)+v(x)] = {}_c\mathrm{D}_x^{\mu} u(x) + {}_c\mathrm{D}_x^{\mu} v(x), \quad . \quad . \quad (1.15)$$

† Davis, *loc. cit.*

$$ {}_cD_x^{\mu} k u(x) = k\, {}_cD_x^{\mu} u(x), \quad . \quad . \quad . \quad . \quad (1.16) $$

$$ {}_cD_x^{\mu}\, {}_cD_x^{\nu} u(x) = {}_cD_x^{\mu+\nu} u(x), \quad . \quad . \quad . \quad . \quad (1.17) $$

where k is a constant and μ and ν are real numbers.

In order to be clear at the outset about the nature of the functions involved in our work, we shall assume that the arbitrary functions to which equations (1.13) and (1.14) will be applied satisfy the hypothesis of the following Theorems II. and III. respectively.

Theorem II.

If $f(x)$ is finite and has only a finite number of discontinuities in the interval $\mathrm{I} \equiv (c \leq x \leq b)$, the function ${}_cD_x^{-\nu} f(x)$ is continuous throughout I, *including the point c, where it vanishes* *.

Theorem III.

If $f(x)$ is $(m+1)$ times differentiable, and if its $(m+1)$ st derivative is finite and has at most a finite number of discontinuities in $\mathrm{I} \equiv (c < x \leq b)$, and if $f(c) = f'(c) = \ldots = f^m(c) = 0$, the function ${}_cD_x^{m+\nu} f(x)$ is continuous throughout I.

The proof of Theorem II. is given elsewhere †, while Theorem III. follows from the fact that if

$$ \phi(x) = \int_c^x \frac{f(t)}{(x-t)^{\lambda}} dt, \qquad (\lambda < 1) $$

where $f(t)$ satisfies the hypothesis of this theorem, then

$$ \phi^{(m+1)}(x) = \int_c^x \frac{f^{(m+1)}(t) dt}{(x-t)^{\lambda}} \ddagger, $$

and hence by Theorem II. is continuous.

2. *The Fundamental Theorems.*

We shall now derive several formulæ each of which can be considered as an integral equation of a certain type together with its solution. It will be seen that the fundamental theorems of Wrinch and Nicholson's paper § are

* We define the symbol $\int_c^c \phi(x) dx$ to mean zero, whatever the nature of $\phi(x)$ may be.

† M. Bôcher, 'An Introduction to the Study of Integral Equations,' Cambridge University Press, London, pp. 4, 5.

‡ *Ibid.*

§ Wrinch and Nicholson, *loc. cit.* pp. 539, 540.

particular cases of our more general formulæ. In the paper just referred to the main results have been obtained by means of Fourier's cosine and sine formulæ, the Bessel integral theorem, and by means of two equations due to Sonine*. Our more general theorems will turn out to be direct consequences of the definitions of generalized integration and differentiation.

By the definition (1.13) and by (1.17) we can write

$$\phi(X) = \frac{d^p}{dX^p}{}_a D_X^{-\sigma+\sigma-p}\phi(X), \quad (0<\sigma<p\,;\ p=1, 2, \ldots). \quad . \quad . \quad . \quad (2.1)$$

Written out explicitly this gives the formula

$$\phi(X) = \frac{d^p}{dX^p}\int_a^X \frac{(X-s)^{\sigma-1}}{\Gamma(\sigma)}\,ds \int_a^s \frac{(s-t)^{p-\sigma-1}}{\Gamma(p-\sigma)}\,\phi(t)\,dt, \quad (0<\sigma<p\,;\ p=1, 2, \ldots). \quad (2.2)$$

Taking out the factor $(-1)^{\sigma-1}$ from under the first integral sign, and $(-1)^{p-\sigma-1}$ from under the second integral sign, we can rewrite the last equation in the form

$$\phi(X) = (-1)^p \frac{d^p}{dX^p}\int_a^X \frac{(s-X)^{\sigma-1}}{\Gamma(\sigma)}\,ds \int_a^s \frac{(t-s)^{p-\sigma-1}}{\Gamma(p-\sigma)}\,\phi(t)\,dt, \quad (0<\sigma<p\,;\ p=1, 2, 3, \ldots). \quad (2.3)$$

Each of the formulæ derived in this section can be rewritten in the form of a Volterra integral equation of the first kind with finite or infinite kernel, together with its solution. From (2.2) we obtain that:

If

$$g(s) = \int_a^s \frac{(s-t)^{p-\sigma-1}}{\Gamma(p-\sigma)}\,\phi(t)\,dt, \quad (0<\sigma<p;\ p=1, 2, \ldots). \quad (2.4)$$

then

$$\phi(X) = \frac{d^p}{dX^p}\int_a^X \frac{(s-X)^{\sigma-1}}{\Gamma(\sigma)}\,g(s)\,ds. \quad . \quad . \quad (2.4')$$

Similarly from (2.3) we have that:

If

$$g(s) = \int_a^s \frac{(t-s)^{p-\sigma-1}}{\Gamma(p-\sigma)}\,\phi(t)\,dt, \quad (0<\sigma<p;\ p=1, 2, \ldots). \quad (2.5)$$

* Sonine, "Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries," *Math. Annalen*, xvi. pp. 1-80 (1880).

then $$\phi(t)=(-1)^p \frac{d^p}{dX^p}\int_a^X \frac{(s-X)^{\sigma-1}}{\Gamma(\sigma)}\, g(s)ds. \quad . \quad (2.5')$$

The nature of the arbitrary functions employed having been specified in the preceding section, we are now free from the necessity of proving any "existence" theorems in connexion with the results of this section.

Changing variables in (2.2) by setting $X = x^n$, $s = y^n$, $t = z^n$, $a = c^n$, we obtain

$$\phi(x^n)=\frac{n^2}{\Gamma(\sigma)\Gamma(p-\sigma)}\frac{d^p}{d(x^n)^p}\int_c^x (x^n-y^n)^{\sigma-1}y^{n-1}dy \times \int_c^y (y^n-z^n)^{p-\sigma-1}z^{n-1}\phi(z^n)dz, \quad (0<\sigma<p\,;\ p=1,\,2,\,3,\,\ldots). \quad (2.6)$$

If for $x^{n-1}\phi(x^n)$ we write $f(x)$, we obtain the following theorem:

THEOREM A.

$$f(x)=\frac{n^2x^{n-1}}{\Gamma(\sigma)\Gamma(p-\sigma)}\frac{d^p}{d(x^n)^p}\int_c^x (x^n-y^n)^{\sigma-1}y^{n-1}dy \times \int_c^y (y^n-z^n)^{p-\sigma-1}f(z)\,dz, \quad (0<\sigma<p\,;\ p=1,\,2,\,3,\,\ldots). \quad (2.7)$$

If in (2.7) we take out the factor $(-1)^{\sigma-1}$ from under the first integral sign and $(-1)^{p-\sigma-1}$ from under the second integral sign, and let $p-\sigma=\nu$, we obtain another theorem:

THEOREM B.

$$f(x)=\frac{(-1)^p n^2 x^{n-1}}{\Gamma(\nu)\Gamma(p-\nu)}\frac{d^p}{d(x^n)^p}\int_c^x (y^n-x^n)^{p-\nu-1}y^{n-1}dy \times \int_c^y (z^n-y^n)^{\nu-1}f(z)dz, \quad (0<\nu<p\,;\ p=1,\,2,\,3,\,\ldots). \quad (2.8)$$

If in Theorem A we let $n=2$, $p=1$, $c=0$, we obtain, since $\Gamma(\sigma)\Gamma(1-\sigma)=\pi/\sin\sigma\pi$,

$$f(x)=\frac{2\sin\sigma\pi}{\pi}\frac{d}{dx}\int_0^x (x^2-y^2)^{\sigma-1}y\,dy \times \int_0^y (y^2-z^2)^{-\sigma}f(z)\,dz, \quad (0<\sigma<1). \quad (2.9)$$

which is one of Wrinch and Nicholson's fundamental theorems *, except that the interval of validity for σ has been extended from $\frac{1}{2} < \sigma < 1$ to $0 < \sigma < 1$. If in Theorem B we let $n = 2$, $p = 1$, $c = \infty$, and if we reverse the directions of integration, we derive the equation

$$f(x) = -\frac{2\sin \nu\pi}{\pi} \frac{d}{dx} \int_x^\infty (y^2 - x^2)^{-\nu} y \, dy \times \int_y^\infty (z^2 - y^2)^{\nu-1} f(z) dz, \quad (0 < \nu < 1). \quad (2.10)$$

which is an extension of the second fundamental theorem in the paper just referred to.

Rewriting our main theorems in the form of integral equations and their solutions, we obtain from (2.7) that :

If

$$g(y) = \int_c^y (y^n - z^n)^{p-\sigma-1} f(z) dz, \quad (0 < \nu < p \; ; \; p = 1, 2, 3, \ldots), \quad (2.11)$$

then

$$f(x) = \frac{n^2 x^{n-1}}{\Gamma(\sigma)\Gamma(p-\sigma)} \frac{d^p}{d(x^n)^p} \int_c^x (x^n - y^n)^{\sigma-1} y^{n-1} g(y) dy. \quad (2.11')$$

Similarly, from (2.8) we have that :

If

$$g(y) = \int_c^y (z^n - y^n)^{\nu-1} f(z) dz, \quad (0 < \nu), \quad (2.12)$$

then

$$f(x) = \frac{(-1)^p n^2 x^{n-1}}{\Gamma(\nu)\Gamma(p-\nu)} \frac{d^p}{d(x^n)^p} \int_c^x (y^n - x^n)^{p-\nu-1} y^{n-1} g(y) dy, \quad (p > \nu \; ; \; p = 1, 2, 3, \ldots). \quad (2.12')$$

If in the last two integral equations and their solutions we let $p = 1$, we obtain the results :

If

$$g(y) = \int_c^y (y^n - z^n)^{-\sigma} f(z) dz, \quad (0 < \sigma < 1), \quad (2.13)$$

then

$$(x) = \frac{n \sin \sigma\pi}{\pi} \frac{d}{dx} \int_c^x (x^n - y^n)^{\sigma-1} y^{n-1} g(y) dy. \quad . \quad (2.13')$$

* Wrinch and Nicholson, *loc. cit.* pp. 539, 540.

If

$$g(y) = \int_c^y (z^n - y^n)^{\nu-1} f(z)dz, \quad (0<\nu), \quad (2.14)$$

then

$$f(x) = -\frac{n \sin \nu\pi}{\pi} \frac{d}{dx} \int_c^x (y^n - x^n)^{-\nu} y^{n-1} g(y)dy. \quad . \quad (2.14')$$

A large class of integral equations occurring in physics are of the form (2.13) and (2.14). We shall state now some of these equations, together with their solutions, which follow from (2.13′) and (2.14′).

The Bateman-Herglotz * equation, which arises in problems on seismology, is usually written in the form

$$\phi(x) = x \int_x^\alpha \frac{f'(t)\,dt}{(t^2 - x^2)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad (2.15)$$

This can be written in the form

$$\frac{\phi(x)}{x} = -\int_\alpha^x \frac{f'(t)dt}{(t^2 - x^2)^{\frac{1}{2}}},$$

which is the particular case of (2.14) for which $n = 2$, $\nu = \frac{1}{2}$. Therefore, by (2.14′),

$$f'(x) = -\frac{2}{\pi} \frac{d}{dx} \int_x^\alpha \frac{\phi(t)dt}{(t^2 - x^2)^{\frac{1}{2}}}.$$

Integrating both sides with respect to x, we obtain the solution of (2.15) as

$$f(x) = -\frac{2}{\pi} \int_x^\alpha \frac{\phi(t)dt}{(t^2 - x^2)^{\frac{1}{2}}} + c, \quad . \quad . \quad . \quad (2.15')$$

where c is a constant. This is the well-known result †.

Abel's equation is of the form

$$f(x) = \int_c^x \frac{u(t)dt}{(x-t)^\nu}, \qquad (0<\nu<1). \quad . \quad (2.16)$$

This is a particular case of (2.13) with $n = 1$, therefore by (2.13′)

$$u(x) = \frac{\sin \nu\pi}{\pi} \frac{d}{dx} \int_c^x \frac{f(s)ds}{(x-s)^{1-\nu}}. \quad . \quad . \quad (2.16')$$

* Jeffreys, 'The Earth,' Cambridge University Press, p. 176 (1924).
† Jeffreys, *loc. cit.*

Other equations, with their solutions, which have found applications in physics, and might therefore be of interest, are the following:

If

$$g(x)=\int_c^x \frac{f(t)dt}{(x^2-t^2)^\nu}, \quad (0<\nu<1), \quad (2.17)$$

then by (2.13′)

$$f(x)=\frac{2\sin\nu\pi}{\pi}\frac{d}{dx}\int_c^x \frac{tg(t)dt}{(x^2-t^2)^{1-\nu}}. \quad . \quad . \quad (2.17')$$

If

$$g(x)=\int_x^c \frac{f(t)dt}{(t^2-x^2)^{1-\nu}}, \quad (0<\nu<1), \quad (2.18)$$

then by (2.14) and (2.14′) for $\sigma=1-\nu$ we obtain

$$f(x)=-\frac{2\sin(1-\nu)\pi}{\pi}\frac{d}{dx}\int_x^c \frac{sg(s)ds}{(s^2-x^2)^\nu}$$

$$=-\frac{2\sin\nu\pi}{\pi}\frac{d}{dx}\int_x^c \frac{sg(s)ds}{(s^2-x^2)^\nu}. \quad . \quad . \quad (2.18')$$

The integral equations corresponding to the formulæ of Theorems A and B with $p=1$ involve the first derivatives in front of the integral sign of the solution. Such equations are met with quite frequently in the literature dealing with the subject of integral equations. The formulæ of our theorems involve, however, the pth derivative (p equal to a positive integer), and might therefore be of some special interest. For this reason we shall rewrite these equations for particular values of p, say $p=2$ and $p=3$.

Setting $p=2$ in (2.7) and noting that

$$\frac{d^2}{d(x^n)^2}=\frac{1}{n^2x^{2n-1}}\frac{d^2}{dx^2}+\frac{(1-n)}{n^2x^{2n-1}}\frac{d}{dx}, \quad . \quad . \quad (2.19)$$

we obtain the equation

$$f(x)=\frac{x^{-n}}{\Gamma(\sigma)\Gamma(2-\sigma)}\left[x\frac{d^2}{dx^2}+(1-n)\frac{d}{dx}\right]$$

$$\times\int_0^x (x^n-y^n)^{\sigma-1}y^{n-1}dy\int_c^y (y^n-z^n)^{1-\sigma}f(z)dz,$$

$$(0<\sigma<2). \quad (2.20)$$

Setting $p=3$ in (2.7), and noting that

$$\frac{d^3}{d(x^n)^3}=\frac{1}{n^3x^{3n-3}}\frac{d^3}{dx^3}+\frac{3(1-n)}{n^3x^{3n-2}}\frac{d^2}{dx^2}+\frac{(1-n)(1-2n)}{n^3x^{3n-1}}\frac{d}{dx}, \quad (2.21)$$

we obtain

$$f(x) = \frac{x^{-2n}}{n\Gamma(\sigma)\Gamma(3-\sigma)}\left[x^2\frac{d^3}{dx^3} + 3(1-n)x\frac{d^2}{dx^2} + (1-n)(1-2n)\frac{d}{dx}\right] \cdot \int_c^x (x^n - y^n)^{\sigma-1} y^{n-1} dy \int_c^y (y^n - z^n)^{2-\sigma} f(z) dz, \quad (0<\sigma<3). \quad . \quad . \quad . \quad (2.22)$$

The expression for $\frac{d^p}{d(x^n)^p}$ can be obtained by means of the following equations * :—If u is a function of x, and x is a function of X, then

$$\frac{d^p u}{dX^p} = A_1\frac{du}{dx} + \frac{A_2}{2!}\frac{d^2u}{dx^2} + \ldots + \frac{A_p}{p!}\frac{d^p u}{dx^p},$$

$$A_k = \frac{d^p x^k}{dX^p} - \frac{kx}{1!}\frac{d^p x^{k-1}}{dX^p} + \frac{k(k-1)x^2}{2!}\frac{d^p x^{k-2}}{dX^p} + \ldots + (-1)^{k-1}kx^{k-1}\frac{d^p x}{dX^p}. \quad . \quad . \quad . \quad (2.23)$$

In our formulæ $x^n = X$.

If in the equation under Theorem A we make the following substitutions: $z/y = \lambda/x$, $dz = \frac{y}{x}d\lambda$, $c = 0$, then we obtain

$$f(x) = \frac{n^2 x^{n-1}}{\Gamma(\sigma)\Gamma(p-\sigma)}\frac{d^p}{d(x^n)^p}x^{n(\sigma+1-p)}\int_0^x \frac{d\lambda}{(x^n-\lambda^n)^{\sigma+1-p}} \int_0^x \frac{f\left(\frac{\lambda y}{x}\right)\frac{dy}{x}}{y^{n(1-p)}\left(\frac{x^n}{y^n}-1\right)^{1-\sigma}}. \quad . \quad (2.24)$$

If in this equation we set $y = \frac{x}{\lambda}\mu$, $dy = \frac{x}{\lambda}d\mu$, then there results the next theorem :

THEOREM A'.

$$f(x) = \frac{n^2 x^{n-1}}{\Gamma(\sigma)\Gamma(p-\sigma)}\frac{d^p}{d(x^n)^p}x^n\int_0^x \frac{\lambda^{n-np-1}d\lambda}{(x^n-\lambda^n)^{\sigma+1-p}} \times \int_0^\lambda \frac{\mu^{n(p-\sigma)}f(\mu)\,d\mu}{(\lambda^n-\mu^n)^{1-\sigma}}, \quad (p = 1, 2, 3\ldots;\ 0<\sigma<p). \quad (2.25)$$

* Goursat-Hedrick, 'Mathematical Analysis,' i. p. 53.

Setting $p=1$, $n=2$, this equation becomes

$$f(x)=\frac{2\sin\sigma\pi}{\pi}\frac{d}{dx}x^{2\sigma}\int_0^x\frac{d\lambda\,\lambda}{(x^2-\lambda^2)^\sigma}\int_0^\lambda\frac{\mu^{2-2\sigma}f(\mu)d\mu}{(\lambda^2-\mu^2)^{1-\sigma}},$$

$$(0<\sigma<1), \quad (2.26)$$

which is another form of an extension of formula A in Wrinch and Nicholson's paper.

If in (2.25) we take out the factor $(-1)^{\sigma+1-p}$ from under the first integral sign and $(-1)^{1-\sigma}$ from under the second integral sign we obtain the

THEOREM B'.

$$f(x)=\frac{(-1)^p n^2 x^{n-1}}{\Gamma(\sigma)\Gamma(p-\sigma)}\frac{d^p}{d(x^n)^p}x^{n\sigma}\int_0^x\frac{\lambda^{n-np-1}d\lambda}{(\lambda^n-x^n)^{\sigma+1-p}}$$

$$\times\int_0^\lambda\frac{\mu^{np-n\sigma}f(\mu)d\mu}{(\mu^n-\lambda^n)^{1-\sigma}}, \quad (0<\sigma<p;\ p=1,2,\ldots). \quad (2.27)$$

Writing equation (2.25) in the form of an integral equation and its solution we have:

If

$$g(\lambda)=\int_0^\lambda\frac{\mu^{n(p-\sigma)}f(\mu)d\mu}{(\lambda^n-\mu^n)^{1-\sigma}}, \quad (0<\sigma<p), \quad (2.28)$$

then

$$f(x)=\frac{n^2x^{n-1}}{\Gamma(\sigma)\Gamma(p-\sigma)}\frac{d^p}{d(x^n)^p}x^{n\sigma}\int_0^x\frac{\lambda^{n-np-1}g(\lambda)d\lambda}{(\lambda^n-x^n)^{\sigma+1-p}}. \quad (2.28')$$

If in (2.28) we set $p=1$, $n=2$, $\sigma=\frac{1}{2}$, then

$$g(\lambda)=\int_0^\lambda\frac{\mu f(\mu)d\mu}{(\lambda^2-\mu^2)^{\frac{1}{2}}},$$

and hence by (2.28')

$$f(x)=\frac{2}{\pi}\frac{d}{dx}x\int_0^x\frac{g(\lambda)\,d\lambda/\lambda}{(x^2-\lambda^2)^{\frac{1}{2}}}.$$

Examples of this equation are already familiar in physics*.

Let us set $p-\sigma=\nu$ in (2.28). Then this equation takes on the form

$$g(\lambda)=\int_0^\lambda\frac{\mu^{n\nu}f(\mu)\,d\mu}{(\lambda^n-\mu^n)^{\nu-p+1}},$$

$$(p>\nu>0\,;\ p=1,2,3,\ldots), \quad (2.29)$$

* Wrinch and Nicholson, *loc. cit.*, p. 544.

and (2.28′) becomes

$$f(x) = \frac{n^2 x^{n-1}}{\Gamma(p-\nu)\Gamma(\nu)} \frac{d^p}{d(x^n)^p} x^{n(p-\nu)} \int_0^\lambda \frac{\lambda^{n-np-1} g(\lambda) d\lambda}{(x^n - \lambda^n)^{1-\nu}}. \quad (2.29')$$

If in (2.29) $p = 1$, $n = 2$, we derive the equation

$$g(\lambda) = \int_0^\lambda \frac{\mu^{2\nu} f(\mu) d\mu}{(\lambda^2 - \mu^2)^\nu}, \quad (0<\nu<1). \quad (2.30)$$

As a solution of this integral equation we have by (2.29′)

$$f(x) = \frac{2 \sin \nu\pi}{\pi} \frac{d}{dx} x^{2-2\nu} \int_0^\lambda \frac{\lambda^{-1} g(\lambda) d\lambda}{(x^2 - y^2)^{1-\nu}}. \quad . \quad (2.30')$$

All integral equations considered in this section were derived by means of a change of variables from the equation (2.2). This equation, however, represents nothing else but the integral equation (2.4), which is of Volterra's first type, together with its solution. Hence it follows that every one of the equations considered here, and also a large number of integral equations occurring in physics, can be derived by means of a change of variables from Volterra's integral equation of the first kind of the particular form

$$\phi(x) = \int_a^x K(x-s) f(s) ds. \quad . \quad . \quad (2.31)$$

The general methods for solving this equation with $K(x-s)$ being a finite or infinite kernel have been studied quite considerably, and are therefore well known. In this paper the author has merely attempted to devise a simple scheme for obtaining the solutions of some equations more readily from their original form, instead of reducing them to the form (2.31) and applying the general method for solving such an equation.

XLII. *On the Fingering of Conical Wind Instruments.* By Eric J. Irons, *Ph.D.**

IN a former paper formulæ for the "acoustical impedance" of various units were derived [1], and in a second paper the results were applied to problems concerning the fingering of wind instruments of cylindrical form [2]; in

* Communicated by the Author.

this note the argument is extended to include instruments of conical shape.

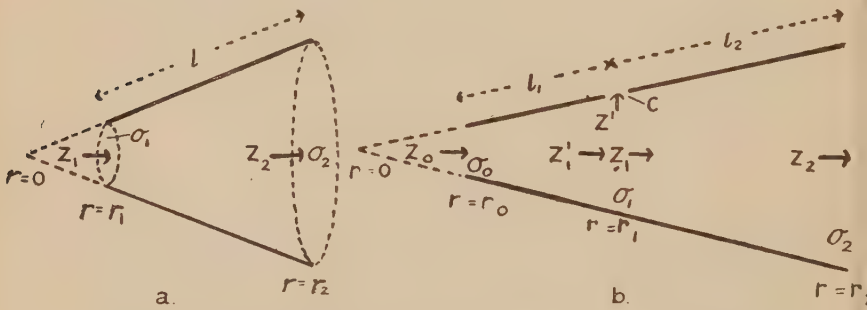
With the previous notation the impedance, Z_1, at the smaller end of the cone shown in fig. *a* is given by

$$Z_1 = -\frac{\beta}{\sigma_1} \cdot \frac{Z_2 \sin k(l-\epsilon_2)/\sin k\epsilon_2 + (\beta/\sigma_2)\sin kl}{Z_2 \sin k(l+\epsilon_1-\epsilon_2)/\sin k\epsilon_1 \cdot \sin k\epsilon_2 + (\beta/\sigma_2)\sin k(l+\epsilon_1)/\sin k\epsilon_1}, \quad . \quad (1)$$

where $\tan k\epsilon_1 = kr_1$ and $\tan k\epsilon_2 = kr_2$, and the impedance due to an orifice is

$$Z' = -\omega^2\rho/c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Consider a conical instrument of the form and dimensions shown in fig. *b*, having a side hole at $r=r_1$ and a mouth at $r=r_0$, and suppose that the lengths involved include any necessary end corrections.



By the methods of the previous papers we have, on applying (1) and (2),

$$Z_2 = 0,$$

$$\therefore Z_1 = -\frac{\beta}{\sigma_1} \cdot \frac{\sin kl_2 \cdot \sin k\epsilon_1}{\sin k(l_2+\epsilon_1)} \quad \text{where} \quad \tan k\epsilon_1 = kr_1,$$

and, as

$$1/Z' = -c/\omega^2\rho,$$

$$1/Z_1' = -c/\omega^2\rho - (\sigma_1/\beta)(\cot k\epsilon_1 + \cot kl_2). \quad . \quad . \quad . \quad (3)$$

Again,

$$Z_0 = -\frac{\beta}{\sigma_0} \cdot \frac{Z_1' \sin k(l_1-\epsilon_1)/\sin k\epsilon_1 + (\beta/\sigma_1)\sin kl_1}{Z_1' \sin k(l_1+\epsilon_0-\epsilon_1)/\sin k\epsilon_0 \sin k\epsilon_1 + (\beta/\sigma_1)\sin k(l_1+\epsilon_0)/\sin k\epsilon_0}, \quad . \quad . \quad (4)$$

where $\tan k\epsilon_0 = kr_0$.

For an *antinodal* source (flute or organ-pipe type) $Z_0=0$, and from (4)

$$1/Z_1' = -(\sigma_1/\beta)(\cot k\epsilon_1 - \cot kl_1),$$

whence by (3)

$$-1/(\cot kl_1 + \cot kl_2) = (\sigma_1/c)k. \quad . \quad . \quad . \quad (5)$$

The frequency, n_0, of the pipe with no side holes open is given by substituting $c=0$ in (5), when $\tan k(l_1+l_2)=0$ and $n_0=a/2(l_1+l_2)$.

If $f=l_1/(l_1+l_2)$, (5) becomes

$$-1\Big/\left(\cot\pi\,.\,\frac{n}{n_0}\,.\,f+\cot\pi\,.\,\frac{n}{n_0}\,.\,\overline{1-f}\right) = \frac{\pi}{(l_1+l_2)}\,.\,\frac{\sigma_1}{c}\,.\,\frac{n}{n_0}. \quad (6)$$

For a *nodal* source (reed type) $Z_0=\infty$, and from (4)

$$1/Z_1' = -(\sigma_1/\beta)(\cot k\epsilon_1 - \cot\,.\,k\,.\,\overline{l_1+\epsilon_0}),$$

whence, by (3),

$$-1/(\cot k\,.\,\overline{l_1+\epsilon_0} + \cot kl_2) = (\sigma_1/c)\,.\,k, \quad . \quad (7.1)$$

or

$$-1/(-\tan k\,.\,\overline{l_1+\delta} + \cot kl_2) = (\sigma_1/c)k, \quad . \quad (7.2)$$

where $\cot k\delta = kr_0$.

The frequency of the pipe with no side holes open is, as before, found by substituting $c=0$ in (7.1), when $\tan k(l_1+l_2+\epsilon_0)=0$, and $n_0=a/2(l_1+l_2+\epsilon_0)$; or, by substituting $c=0$ in (7.2), when $\tan k(l_1+l_2+\delta)=\infty$ and $n_0=a/4(l_1+l_2+\delta)$. Putting $f=(l_1+\epsilon_0)/(l_1+l_2+\epsilon_0)$ in (7.1) and $f=(l_1+\delta)/(l_1+l_2+\delta)$ in (7.2), these equations reduce to

$$-1\Big/\left(\cot\pi\,.\,\frac{n}{n_0}\,.\,f+\cot\pi\,.\,\frac{n}{n_0}\,.\,\overline{1-f}\right) = \frac{\pi}{(l_1+l_2+\epsilon_0)}\,.\,\frac{\sigma_1}{c}\,.\,\frac{n}{n_0}, \quad . \quad (8.1)$$

and

$$-1\Big/\left(-\tan\frac{\pi}{2}\,.\,\frac{n}{n_0}\,.\,f+\cot\frac{\pi}{2}\,.\,\frac{n}{n_0}\,.\,\overline{1-f}\right) = \frac{\pi}{2(l_1+l_2+\delta)}\,.\,\frac{\sigma_1}{c}\,.\,\frac{n}{n_0} \quad (8.2)$$

respectively.

If we make $r_2 \to \infty$, $r_1 \to \infty$, and $r_0 \to \infty$ in such a manner that $(r_2 - r_1)$ and $(r_1 - r_0)$ remain finite and equal to l_2 and l_1 respectively, σ_1 becomes identical with σ, the area of cross-section of the resulting tube, and, as $\tan k\epsilon_0 = kr_0$, ϵ_0 becomes $\pi/2k$. Equations (5) and (7) then become

$$-1/(\cot kl_1 + \cot kl_2) = (\sigma/c)k \quad . \quad . \quad . \quad . \quad (9)$$

and

$$-1/(-\tan kl_1 + \cot kl_2) = (\sigma/c)k \quad . \quad . \quad (10)$$

respectively—equations previously obtained for a cylindrical instrument [2].

Moreover, with two or more side holes open a cylindrical instrument may be considered as one pipe extending from the mouth to the first side hole and another pipe extending from this hole to the second hole (at which an antinode is supposed to form). If we may assume that the same approximation is valid for a conical instrument, then equations (6) and (8) represent, in addition, the behaviour of instruments with a number of side holes open if the two nearest to the mouth are situated at $r = r_1$ and $r = r_2$ (fig. *b*).

As regards equation (6) values of n may be obtained in terms of n_0 by plotting values of the left-hand side of (6) against n/n_0 for various values of f, plotting values of the right-hand side against n/n_0, and marking the intersections of the resulting curves and straight lines. This has been done in fig. 2 (Ref. (2)) *, and it is to be noted that if l_1 and l_2 are measured parallel to either side of the cone and the value of the cross-section employed is that at the hole nearest the mouth this figure may be used for the determination of the resonant frequencies of a conical instrument "open" at both ends.

For a conical horn having a source of the reed type f is a function of ϵ_0 or δ, and therefore of k. In consequence, the graphical solution which serves for equation (6) (Ref. (2), fig. 2) may only be used if $k\epsilon_0$ is so small that $\epsilon_0 = r_0$ (eq. (8.1)), and the conditions become approximately equivalent to those of a horn with an antinodal source very near the apex. Alternatively, if δ is negligibly small (eq. (8.2)), the conditions are such that $\cot k\delta \to \infty$ and $kr_0 \to \infty$ (*i.e.*, the cone approximates to a cylinder), and the graph given in the previous paper [2] under fig. 3 may, with due regard to the interpretation of f, be employed. For intermediate cases (7) may be solved by successive approximations.

* On page 25, line 39, and page 26, line 10, for fig. 3 read fig. 2.

As actual instruments are only approximately conical in shape, the treatment of resonance in horns of other shapes will form the subject of a later communication.

References.

(1) Irons, Phil. Mag. ix. p. 346 (1930).
(2) Irons, Phil. Mag. x. p. 16 (1930).

XLIII. *Magnetism and Electrodynamics.* *By* Irene E. Viney *.

1. *Introduction.*

THE assumption of the existence of free magnetism was introduced into electrodynamic theory by Heaviside † and Hertz ‡. It has been made spasmodically on various occasions since, and has recently assumed some importance in connexion with certain developments of the quantum theory and wave mechanics. In this latter connexion it was first introduced by E. T. Whittaker §. As a result of Whittaker's work Lorentz ‖ examined the influence of the assumption on the general relations of the electromagnetic theory, and, although he gives no analysis, he draws the following conclusion :—"The theory takes the simplest form, when it is assumed that there are not only electric charges, but also magnetic ones—accumulations of positive or negative magnetism. By the introduction of these into the fundamental equations the parallelism between the electric and magnetic quantities can be clearly brought out." However, at the conclusion of the second part of the paper dealing directly with Whittaker's work, he states that when terms representing the existence of free magnetism are included in the fundamental equations "the equations, though forming a consistent system, do not allow of the establishment of the variation theorems of the kind of Hamilton's Principle. In this Principle we are concerned with the difference between the potential and kinetic energy, so that in the equations the two energies do not occur in the same way. Now if there

* Communicated by Prof. G. H. Livens.
† 'Electrical Papers,' i. pp. 441–452 (1892).
‡ 'Electric Waves,' p. 217 (1893).
§ Proc. R. S. Edinb. xlii. p. 129 (1922).
‖ *Proc. K. Akad. Amst.* xxv. p. 414 (1923).

are only electric charges, we can, as is well known, arrive at an equation of the Hamiltonian form, in which $\frac{1}{8\pi}E^2$ is the potential, and $\frac{1}{8\pi}H^2$ the kinetic energy. If there are only magnetic charges, there is a similar formula, in which, however, the electric and magnetic energy have changed their parts. It is clear that it must be difficult to combine the two theories into one."

As it seemed possible that the difficulty seen by Lorentz was really of a more fundamental kind than he recognized, and might appear at an earlier stage of the theory—in spite of the satisfactory nature of his conclusions—it was suggested to me that an attempt might be made to rewrite the earlier stages of the theory on the assumption of the existence of free magnetism, and the main conclusions of the attempt are summarized in the present note. It appears that it is quite impossible to obtain a completely consistent formulation of the magnetic side of the theory without the assumption that div $B = 0$ everywhere in the field, and, further, that the parallelism in the relations of the electric and magnetic fields is not nearly so satisfactory as Lorentz and other writers seem to assume.

2. *The Statical Magnetic Field without Currents.*

We shall first of all briefly summarize the usual and familiar relations of the magnetostatic field, regarding it mathematically as the complete analogue of the electrostatic field. We assume the existence of free magnetism or magnetic charge—of both signs—and then regard the physical element of magnetic matter as a doublet or bipolar particle, just as the element of a dielectric is assumed to be an electric bipole. The specification of the field then involves certain scalar and vector quantities analogous to those of the electrostatic theory, viz :—

The Magnetic Force H is defined generally as the magnetic force intensity per unit of magnetism, or, if a small element of magnetism $d\mu$ be placed at a point, then $Hd\mu$ is the force acting on it.

The Potential Function ϕ is defined so that $\phi d\mu$ is the work done by the magnetic forces on a small element of magnetic charge $d\mu$ when carried from that point to infinity, where, by convention, the value of ϕ is zero.

The Magnetization I is defined as the magnetization per unit volume of the field; it is derived by an averaging process as the vectorial sum of the moments of all the bipolar particles, which go to make up the volume element dv, so that $\mathrm{I}dv$ is the resultant moment of the element, I being the intensity of magnetization.

The Magnetic Induction B is defined so that $\mathrm{B}_n df$ represents the magnetic displacement across the area df, B_n denoting the component of B, normal to df. This is the analogue of the electric displacement D, but by convention it is defined so as to include an extra 4π factor, such that

$$\mathrm{B} = \mathrm{H} + 4\pi\mathrm{I}.$$

The Magnetic Density μ is defined as the amount of magnetism or magnetic charge per unit volume at each point in the field.

Obviously the quantities defined by H, ϕ, I, B, μ are analogous to the quantities E, ϕ, P, D, ρ of the dielectric theory; and just as the question of defining E and ϕ at points inside a polarized dielectric meets with some difficulty, so we have here exactly similar difficulties with regard to the definition of H and ϕ at points in the field which are internal to magnetizable media. The usual procedure is to neglect the local contributions to the value of the functions H and ϕ, so that by the ordinary conventions we can still write

$$\phi_{\mathrm{P}} = \int_{(\mathrm{P})}^{\infty} (\mathrm{H}ds),$$

where ϕ_{P} denotes the value of the function ϕ at the point P in the field. It follows then that

$$\mathrm{H} = -\operatorname{grad}\phi$$

at all points of the field.

The potential function ϕ_1 at any point (x_1, y_1, z_1) in the field is given by

$$\phi_1 = -\int_v (\mathrm{I}\nabla_1)\frac{dv}{r} + \int_v \mu\frac{dv}{r},$$

where $r^2 = (x_1-x)^2 + (y_1-y)^2 + (z_1-z)^2$, and the volume integrals are taken throughout the whole volume of the magnet and of all the magnetic charges.

Poisson's transformation of the first integral then gives

$$\phi_1 = \int_f \mathrm{I}_n\frac{df}{r} + \int_v (\mu - \operatorname{div}\mathrm{I})\frac{dv}{r},$$

the first integral being a surface integral over the surface of each magnet, surfaces of discontinuity inside the magnets being excluded.

As an equivalent to Gauss's theorem in electrostatics we have analogously the same relation between the magnetic force and magnetic charge which, expressed mathematically, is given by

$$\int_f \mathrm{H}_n df = 4\pi \int_v \mu_0 dv,$$

where $\mu_0 = \mu + \mu' = \mu - \mathrm{div\ I}$ is the density of the total magnetic charge at any point in the field. Since the relation must hold for any surface f enclosing a volume v, it follows that

$$\mathrm{div\ H} = 4\pi\mu_0,$$

and therefore that

$$-\nabla^2\phi = 4\pi\mu_0$$

at each point of space.

The relation

$$\mathrm{div\ H} = 4\pi(\mu + \mu') = 4\pi(\mu - \mathrm{div\ I})$$

is equivalent to

$$\mathrm{div}\,(\mathrm{H} + 4\pi\mathrm{I}) = 4\pi\mu,$$

and the vector $\mathrm{H}/4\pi + \mathrm{I}$ is, by analogy, defined as the magnetic displacement or induction vector $\mathrm{B}/4\pi$; so that

$$\mathrm{B} = \mathrm{H} + 4\pi\mathrm{I},$$

as before. H represents the æthereal magnetic induction and is present at all points of the field, while 4πI represents the true induction, which is, however, present only at points in the magnetic media.

3. *The Statical Field with Linear Currents.*

So far we have met with no difficulty in the formulation of our theory. It is when we begin to introduce currents that difficulties arise. In a statical field all electric and magnetic phenomena are entirely independent of each other, but in a dynamical theory the two sets of phenomena are interrelated ; an electric current produces in its neighbourhood a magnetic field, and *vice versa*, and there are two fundamental relations interconnecting the two sets of phenomena. The first is Ampère's law, according to which the magnetic field produced by the current is equivalent to that of a magnetic shell placed on any surface bounded by

the current circuit (which is assumed closed), the direction of magnetization following a right-handed screw relation to the direction of circulation of the current, and the strength of the shell being proportional to that of the current, *i. e.*,

$$m = \mathrm{J}/c,$$

where m is the moment of the magnetic shell, J is the strength of the current, and c is a constant (the velocity of light in the usual absolute units).

The total potential at any point of the general field with magnetism, magnetic media, and, say, one current would then be effectively

$$\phi = \int \mu \frac{dv}{r} - \int \mathrm{div\,I} \frac{dv}{r} + \frac{\mathrm{J}}{c} \int (n\nabla) \frac{df}{r},$$

whence it follows that ϕ is cyclic, with a constant $\frac{4\pi \mathrm{J}}{c}$, with respect to the current, and satisfies the differential equation

$$\nabla^2 \phi = -4\pi(\mu - \mathrm{div\,I})$$

at all points of space as before.

The magnetic force H, defined as the negative gradient of ϕ, still satisfies the condition

$$\mathrm{div\,H} = 4\pi(\mu - \mathrm{div\,I}),$$

so that

$$\mathrm{div\,(H} + 4\pi \mathrm{I}) = 4\pi\mu,$$

which gives

$$\mathrm{div\,B} = 4\pi\mu,$$

as before.

The second relation of the theory is due to Faraday; it states that the electromotive force on any closed current in the field is proportional to the rate of diminution of the total normal induction through the circuit. Here arises our first difficulty. If, for the moment, we take F as the induction vector, then the normal induction through a circuit in Faraday's sense is

$$\int \mathrm{F}_n df$$

taken over any barrier sheet abutting on the circuit. But such an expression has no meaning unless it is independent of the particular surface taken, and this requires

$$\mathrm{div\,F} = 0.$$

But none of our vectors satisfies this condition. In the older and more familiar form of the theory, without free magnetic

charge, the induction vector $B = H + 4\pi I$ did actually satisfy this relation, and that is why this vector was chosen by Faraday in his statement of the law of induction; but if there is free magnetism

$$\operatorname{div} B = 4\pi\mu,$$

and we cannot therefore take $F = B$. But $F = H$ is even more impossible, since by definition

$$\operatorname{div} H = 4\pi(\mu + \mu'),$$

and μ, μ' cannot both be zero if there are magnetizable or magnetic substances about.

As Faraday's law involves only the time rate of change of the induction, it has been suggested* that it might be assumed that B is a composite vector, containing a dynamical part, whose divergence is always zero, and a statical part, which represents the contribution of the free magnetism. This might be good enough in a purely statical theory, but one is forced to ask what would happen if the free magnetism began to move—as it is bound to do sooner or later. Another suggestion due to Lorentz †, which in effect comes to the same as the above, is made in the following words:—"If it be assumed that the solenoidal distribution of B replaces permanent magnetism so that for every closed surface in the field the integral

$$\int B_n df$$

has a constant value, in general different from zero, then it would still be necessary to add a further relation between B and H; a new vector B^p could be introduced in such a way that

$$\dot{B}^p = 0,$$

and that for each closed surface

$$\int B_n^p df = \int B_n df.$$

If, further, we put

$$B - B^p = B',$$

then we derive

$$\operatorname{div} B' = 0,$$

and substituting B′ for B in the fundamental equation

$$\operatorname{curl} E = -\frac{1}{c}\dot{B},$$

* R. Gans, *Encykl. der Math. Wiss.* Bd. v. (2) Art. 15, pp. 292, 338 (1906).

† *Ibid.* Art. 13, p. 101 (1903).

we see that the original relation between B and H is now transformed into an exactly similar one between B′ and H. In other words, in the case of permanent magnetism μ we can superimpose its field on that of the ordinary one with $\operatorname{div} B = 0$, but any such additional field is restricted to be constant in time." This appears to result in a complete elimination of any connexion between the field of the magnetic charges and the ordinary field of the theory. What then is the point of introducing free magnetism into the theory if it has no interaction with other types of magnetism?

The general question is to a certain extent settled by another familiar result. The usual formula for the energy of a current in a magnetic field makes it equal to the energy of an equivalent magnetic shell in the same field. If now F′ is used to denote the general force vector of the field, this energy is, in the usual way,

$$-\frac{J}{c}\int F_n' df,$$

which again requires that $\operatorname{div} F' = 0$. Thus, although our previous argument leaves it rather uncertain as to what vector we are to take as the induction, the present one is quite decisive that the force vector must satisfy the equation

$$\operatorname{div} F' = 0,$$

so that, in the form of the theory now under review, both H and B prove to be quite unsuitable as vectors of either force or induction. It follows that to obtain any consistency in the general relations of the theory we must abandon the dielectric analogy and assume

(1) That the old induction vector is the true force vector of the field, and interpret Faraday's law accordingly.

(2) That nothing in the nature of free magnetism can or ever does exist, so that $\operatorname{div} B = 0$ everywhere in the field.

There is really very little difficulty in accepting (1) in view of the fact that the two vectors B and H are identical in free space and that they differ from one another only inside magnetic matter, where the evaluation of the force presents very serious difficulties, and only results in an uncertain value which might be B as well as it might be H.

Moreover, there is definite experimental evidence* which points to the necessity of interchanging B and H. The Hall

* Livens, Phil. Mag. xxxix. p. 676 (1920).

and allied effects and certain magneto-optical effects display certain so-called irregularities with such substances as iron and nickel, which are entirely removed if the vector B instead of the vector H is taken as the effective magnetic force.

4. *The Mechanical Relations of the Field.*

The foregoing discussion may perhaps be regarded as sufficiently conclusive, but there are further arguments based on the considerations of the energy of the field which point to the same conclusions ; and it is here in particular that the analogy between the electric and magnetic phenomena breaks down completely.

To determine an estimate for the total energy associated with the statical field we can proceed in an elementary manner. If the force intensity H produces an additional virtual displacement $\frac{1}{4\pi}\delta B$ throughout the small volume dv, then the work done during the displacement is equal to

$$\frac{1}{4\pi}\int(H\,.\,\delta B)dv$$

taken throughout the entire volume of the field. This expression can also be written

$$\frac{1}{8\pi}\int\delta H^2\,.\,dv+\int(H\delta I)\,.\,dv,$$

and in this expression the second term represents the increase in the internal energy of the magnetized media, and the first therefore represents the increase in the æthereal energy.

A more technical calculation of the energy, also based on the electrostatic analogy, can be made in the following way:—The energy can in fact be measured by the work done in bringing the elements "which give rise to the field" from infinite dispersion. If we imagine the field to arise from a distribution of permanent magnetism I_0 at any place, and also of free magnetic charge of density μ, the work required to bring up additional charge $\delta\mu$ in the presence of the magnetizable media is

$$\int\phi\,.\,\delta\mu\,.\,dv,$$

while that required to bring up δI_0 is

$$-\int(H\delta I_0)dv,$$

so that in all, if the total potential energy of the field is W,

$$\delta W = \int (\phi \delta \mu) dv - \int (H \delta I_0) dv,$$

which, by the usual argument, is also equal to

$$\int \left\{ \frac{1}{4\pi} (H \delta B) - (H \delta I_0) \right\} dv = \frac{1}{8\pi} \int \delta H^2 dv + \int (H \delta I_i) dv,$$

or to

$$\frac{1}{4\pi} \int [\delta(BH) - \tfrac{1}{2} \delta B^2 + 16\pi^2 (I \delta I)] dv + \int (H \delta I_i) dv,$$

where $I = I_i + I_0$ is the total magnetization, being the sum of the induced and permanent magnetization intensities.

The last integral in each of these forms represents the energy stored in the magnetic media on account of the polarization induced in them. The remaining part then represents the energy stored up in the magnetic field in the æther. This latter may therefore be assumed to be present with a density

$$\frac{1}{8\pi} H^2 \quad \text{or} \quad \frac{1}{4\pi} [(BH) - \tfrac{1}{2} B^2 + 8\pi^2 I^2],$$

this second form being given for a reason which will subsequently appear.

The alternative theory, which regards B as the fundamental force vector, gives, similarly,

$$\delta W = \int [(\phi \delta \mu) - (B \delta I_0)] dv,$$

which is also equal to

$$\frac{1}{4\pi} \int (H \delta B) dv - \int (B \delta I_0) dv$$

$$= \frac{1}{8\pi} \int \delta H^2 dv + \int (H \delta I_i) dv - 4\pi \int (I \delta I_0) dv,$$

and this, by the same argument, is equal to

$$\frac{1}{4\pi} \int [-\tfrac{1}{2} \delta B^2 + \delta(BH) + 4\pi (B \delta I_i)] dv,$$

which expression gives

$$\frac{1}{4\pi} [(BH) - \tfrac{1}{2} B^2]$$

for the æthereal density; except for the absence of the local term in I, this expression is the same as that obtained above in the old theory.

Collecting these results together we see that the density of the potential energy associated with a given field without currents can assume either of the following equivalent forms :—

1. With H as the force vector :

$$\text{(I.)} \quad \frac{1}{8\pi}H^2 + \int(H\delta I_i),$$

$$\text{(II.)} \quad \frac{1}{4\pi}(BH) - \frac{1}{8\pi}B^2 + 2\pi I^2 + \int(H\delta I_i).$$

2. With B as the force vector :

$$\text{(I.)} \quad \frac{1}{8\pi}H^2 - 4\pi\int(I, \delta I_0) + \int(H\delta I_i),$$

$$\text{(II.)} \quad \frac{1}{4\pi}(BH) - \frac{1}{8\pi}B^2 + \int(B\delta I_i);$$

the difference between the forms being merely due to the presence of local terms.

The two forms (I.) and (II.) of either 1 or 2 as here given are exact mathematical equivalents, being derived from each other by means of the relations

$$B = H + 4\pi I,$$
$$I = I_i + I_0.$$

In the case of $\mu = 0$, *i. e.*, with div B = 0, everywhere, the forms for the potential energy density become

3. With H as the force vector :

$$\text{(I.)} \quad -\int(H\delta I_0) = -\frac{1}{4\pi}\int(H\delta B) + \frac{1}{8\pi}H^2 + \int(H\delta I_i),$$

$$\text{(II.)} \quad \frac{1}{4\pi}\int(B\delta H) - \frac{1}{8\pi}B^2 + 4\pi\int(I\delta I) + \int H\delta I_i).$$

4. With B as the force vector :

$$\text{(I.)} \quad -\int(B\delta I_0) = -\frac{1}{4\pi}\int(H\delta B) + \frac{1}{8\pi}H^2 - 4\pi\int(I\delta I_0) + \int(H\delta I_i),$$

$$\text{(II.)} \quad \frac{1}{4\pi}\int(B\delta H) - \frac{1}{8\pi}B^2 - \int(B\delta I_i).$$

There is, however, a special point in connexion with these last results, viz., 3 and 4, which must be mentioned. In the static field in this case we have div B = 0, and H is derived from an acyclic potential function, and thus, by the usual argument,

$$\int(\mathrm{HB})dv = \int(\mathrm{H}\delta\mathrm{B})dv = \int(\mathrm{B}\delta\mathrm{H}) = 0$$

when integrated over the whole field ; it follows, that if we omit the first term in each of the expressions (I.) and (II.) of 3 and 4 we shall have a form for the energy which gives precisely the same total when taken over the whole field ; we have then as equivalent energy densities, which give the correct totals,

$$\text{I.} \quad \frac{1}{8\pi}\mathrm{H}^2 + \int(\mathrm{H}\delta\mathrm{I}_i),$$

and

$$\text{II.} \quad -\frac{1}{8\pi}\mathrm{B}^2 + \int(\mathrm{B}\delta\mathrm{I}_i),$$

or modified by the addition of local terms in I. if B is the force vector, and in II. if H is the force vector.

When a current is added to the system, and built up with the rigid magnetism, it adds an additional term to the energy density which, by the usual argument, is equal to

$$-\frac{1}{4\pi}\int(\mathrm{B}\delta\mathrm{H}),$$

and this provides us with the decisive test of the theory ; in fact we see that if we add this to the expressions 3 (I.) and 4 (I.) for the potential energy density with div B = 0 (*i. e.*, $\mu = 0$), then in either case it gives for the leading terms the expression

$$-\frac{1}{4\pi}(\mathrm{BH}) + \frac{1}{8\pi}\mathrm{H}^2 + \int(\mathrm{H}\delta\mathrm{I}_i),$$

whilst if it is added to the corresponding expressions (II.) they reduce essentially to

$$-\frac{1}{8\pi}\mathrm{B}^2 + \int(\mathrm{B}\delta\mathrm{I}_i),$$

which is exactly the same as the form (II.), when

$$\int(\mathrm{B}\delta\mathrm{H}) = 0.$$

It follows that the form for the density

$$-\frac{1}{8\pi}\mathrm{B}^2 + \int(\mathrm{B}\delta\mathrm{I}_i)$$

will serve whether there is a current present or not, and for this reason it appears to be the most suitable form to take. Without artificial manipulation it is impossible to throw the density (I.) into a form which is so simple and natural.

Further, if we introduce this additional energy into the expressions 1 and 2 when $p \neq 0$ we encounter still greater complications ; we get, in fact, as the main æthereal terms in the two cases I. and II. the formulæ

$$-\frac{1}{4\pi}\int(\mathrm{B}\delta\mathrm{H}) + \frac{1}{8\pi}\mathrm{H}^2,$$

and

$$\frac{1}{4\pi}\int(\mathrm{H}\delta\mathrm{B}) - \frac{1}{8\pi}\mathrm{B}^2$$

respectively, which, of course, reduce correctly when there is no current, but which in the general case are almost impossible to reconcile with a reasonable physical theory. The argument, then, is almost overwhelmingly in favour of the form 4 II., and therefore against the existence of magnetism as a free entity in the field.

It is, perhaps, as well to emphasize that it is the general nature of our reasoning which enables us to argue so conclusively about our results. It shows, for example, that the first term in our formulæ (II.), relating to the statical field without free magnetism or currents, viz.,

$$-\left\{-\frac{1}{4\pi}\int(\mathrm{B}\delta\mathrm{H})\right\}$$

corresponds precisely to the removal from the density of that part which would be due to a current, if such were present, whereas no such interpretation is possible for the first term, viz.,

$$-\left\{-\frac{1}{4\pi}\int(\mathrm{H}\delta\mathrm{B})\right\},$$

in the formulæ (I.), which only in the case of a linear law of induction is actually equal to the above value. It is, in fact, the free use of the linear law of induction that has prevented the anomalies in the usual argument from appearing so vividly before. Of course, if we are to take the form (II.) it is natural to take it in the simplest case obtained, when B is the force vector, and the æthereal contribution to the density

$$-\frac{1}{8\pi}\mathrm{B}^2$$

also suggests the fundamental interpretation of B in this way.

The only remaining point relates to the question of sign* ; the form

$$\frac{1}{8\pi}H^2+\int(H\delta I_i)$$

is the expression for the energy density which is most generally used for both statical and dynamical considerations by practically all writers on electrodynamic theory, and without any very disastrous consequences.

For slowly varying and therefore practically steady motions the general Lagrangian dynamical equations of motion show that the potential and kinetic energies (V and T) of the system enter into the determination of the forces in different ways. In fact, in such cases we know that the force Q_s is practically equal to

$$Q_s=\frac{\partial T}{\partial q_s}-\frac{\partial V}{\partial q_s}$$

(q_s being a typical generalized coordinate of the system), so that it is obtained from the sum of the positive gradient of the kinetic energy and the negative gradient of the potential energy. Thus, if we are to regard the potential energy of the statical case as kinetic energy in the dynamical case, and *vice versa*, the sign attributed to the energy must be reversed as we pass from the consideration of one type of energy to the other, in order that the correct sign of the force may be preserved. Thus, if, as is customary, we want to take magnetic energy as kinetic energy, then we should take the æthereal density as $+\frac{1}{8\pi}B^2$, which, of course, is practically identical in all cases to which the theory can be applied, to the usual form $\frac{1}{8\pi}H^2$. The latter form is the one used by most writers on this subject; it is obtained, however, originally as potential energy by means of a potential function, and then regarded without change of sign as kinetic energy.

5. *Summary.*

In conclusion, then, we can say that:

(I.) The only consistent formulation of the dynamical relations of the magnetic field is obtained by using B, and not H, as the magnetic force vector. That B and not H should be regarded as the fundamental vector of the magnetic

* Livens, 'The Theory of Electricity,' ed. ii. p. 187 (1926).

field was realized in the first place by Larmor*, and his views were further substantiated by Livens†. The present analysis, which shows the results of the attempt to introduce magnetic matter into the theory, perhaps brings out more clearly the fundamental reasons for taking B and not H as the magnetic force vector.

(II.) Taking B as the force implies the non-existence of free magnetism. While purely statical considerations of the magnetic phenomena do not clearly reveal the fact that free magnetism must be non-existent, conclusive evidence for its non-existence is soon derived when we consider the mechanical relations of a field surrounding a system of currents. It follows that notions relating to the existence of free magnetism and of magnetic currents, as have been introduced in the first place by Heaviside‡ and later by Whittaker§ and others, into electromagnetic theory, can now be eliminated from the analysis, as being completely inconsistent with the fundamental basis of the theory. If they are to be retained a complete recast of the whole theory will be necessary, and there are at present no indications of the lines along which this can be made.

(III.) The idea that the relations of the magnetic field can be formulated as a complete analogue of the electrostatic field is wholly incompatible with a logical formulation of the relations of electromagnetic theory.

XLIV. *Colour Vision.* *By* R. A. Houstoun, *D.Sc., Lecturer on Physical Optics in the University of Glasgow* ||.

THE majority of physicists at present believe that there are three independent systems of receptors in the retina, one sensitive to red, one sensitive to green, and one sensitive to blue or violet. These are distributed in much the same way as the coloured starch-grains in a Lumière colour plate, and consist of cones with nerve fibres attached to them. When the red system is excited a

* Proc. Roy. Soc. lxxi. p. 229 (1904).
† Proc. Roy. Soc. xciii. A, p. 21 (1916).
‡ 'Electrician,' p. 306 (1885).
§ Proc. Roy. Soc. Edinb. xlvi. [12] p. 120 (1925); see also *ibid.* xlii. p. 129 (1922).
|| Communicated by the Author.

sensation of red is aroused, and when the green and blue systems are excited sensations of green and blue are aroused. The three systems stand on exactly the same footing, and the resultant colour depends on which of the sensations preponderate.

But whenever an attempt is made to work out this view mathematically it is seen to be impossible. In fig. 1 let the abscissæ represent the logarithm of the intensity of the light falling on the retina and the ordinates the magnitude of the sensation aroused. According to my view the curve is of the form

$$y = \int_{-\infty}^{x} e^{-x^2/2} dx.$$

According to the older view it is a straight line cutting the axis at the threshold of vision, but for the purpose of the

Fig. 1.

argument it is immaterial which view we adopt, and in any case at ordinary working intensities the integral is nearly a straight line. It follows from the work of König and Brodhun on Weber's law, and its verification by Jas. F. Shearer and myself, that the curve is the same for all colours for the light-adapted eye.

Let us suppose that the eye is stimulated with monochromatic red light, and that the logarithm of the intensity in metre-candles is represented by OQ; then the sensation is PQ. Let us suppose that the eye is stimulated with monochromatic green light up to Q; the sensation is again PQ. Suppose the intensity of the red diminished to one half; then the sensation is TV where VQ = log 2. Suppose the intensity of the green diminished to one half; the sensation is again TV. Let the red and green be applied simultaneously at half intensity; they produce a sensation of yellow equal to 2TV, but the stimulus is equivalent to a monochromatic yellow of intensity OQ.

Hence the sensation should be PQ. Thus PQ=2TV for all positions of Q, which is absurd. Putting the argument in words, we cannot reason from equality of stimuli to equality of sensations unless the sensations are proportional to the stimuli *measured in metre candles* : this is quite out of the question.

In the above reasoning we have assumed that monochromatic green light arouses the sensation of green alone. According to the view prevalent it acts to almost the same extent on the red sensation. But this does not affect the principle involved, and I have preferred to leave the argument in its simple form.

Let us state the conditions which a theory of colour vision must satisfy :

(1) The sensation of light is mathematically a function of three variables. All sensations can be produced by combining three stimuli.

(2) Although the sensation is a function of three variables, it cannot be the sum of three separate sensations of brightness, as the preceding reasoning shows. It seems reasonable to assume one sensation of brightness and two sensations of colour, or one sensation of colour and one sensation of saturation.

(3) The colours red and green and yellow and blue are connected in pairs. This is required by the great body of tradition connected with Hering's theory.

(4) Yellow, according to the psychologists, is a fundamental sensation ; we cannot see the red and green in it. Here we are on less certain ground, because for me spectrum violet has always been a fundamental sensation, and the psychologists appear to regard it as composite ; but the case of yellow is, I think, beyond doubt.

(5) Anatomically there is no evidence for three photochemical substances or three nerve systems. But there are two nuclear layers and two reticular layers in the retina.

It is in this last condition that I think the solution lies. According to the usual view the nervous impulse originates in the layer of the rods and cones, passes through the outer nuclear layer, crosses a synapse in the outer reticular layer, passes through the inner nuclear layer, and then crosses a synapse in the inner reticular layer. There are

thus two synapses in the visual path. But as far as I can make out from inspection of the drawings and conversation with workers in this field it is impossible to trace the single fibres with certainty. An alternative interpretation is thus equally probable, namely, that there are two different kinds of visual path, those with a synapse in the outer reticular layer and those with a synapse in the inner reticular layer. The distance between the two layers is so small that a separate neurone seems hardly necessary to carry the impulse from the one to the other. I would suggest, with all the ignorance of a physicist, that the cells are arranged in two layers, so that the cones can pack closer.

Let us assume that there are two different kinds of path. To satisfy the requirements of Hering's theory we have to make a further assumption, namely, that after the synapse each path transmits only two kinds of impulse, or, in other words, there are red-green fibres which transmit either red or green, and blue-yellow fibres which transmit either blue or yellow. If R and P denote respectively the number of red-green fibres in the red and green states and B and Y the number of blue-yellow fibres in the blue and yellow states, since the number of fibres in each class is the same,

$$R+P=B+Y.$$

As the four variables are connected by this relation, only three of them are independent. Thus the laws of colour mixing are satisfied.

It will conduce to clear ideas if we trace the process from the start. Light of a certain frequency falls upon a photochemical substance, and electrons are ejected according to the equation of the quantum theory,

$$h\nu=\tfrac{1}{2}mv^2.$$

The velocity of the electron depends on the frequency of the light and the number ejected on the intensity of the light. These electrons set up pulses in the cones. R. S. Lillie has described a nerve model which imitates very closely the propagation of the pulses. If an iron wire is dipped into a strong solution of nitric acid its surface becomes "passive," *i.e.*, becomes covered by a sheath of oxide which protects it from the further action of the acid. If the wire is then immersed in a weaker solution of the acid and its surface touched with a zinc rod, the sheath is

destroyed at the point of contact and a pulse characterized by effervescence travels rapidly along the wire. The shell reforms behind the pulse, and after a short interval of time the wire is in a condition to transmit another pulse. Now the time of recovery will depend to some extent on the velocity of the electrons, *i.e.*, on the discharging potential. We think therefore of a train of pulses of variable frequency approaching the synapse from the cone, the frequency varying according to the colour of the incident light.

Beyond the synapse there are only two possible frequencies, *e.g.*, in the case of the red-green fibres the fibre must be either in the red or the green state. Perhaps the analogy of an organ pipe may be helpful. If we blow it gently, it sounds the fundamental; but if we blow harder, it gives the first harmonic.

Thus the light process and the colour process are quite separate. One takes place in the layer of the rods and cones and the other in the reticular layers. If the light has a certain intensity the same fibres are stimulated, no matter what is the colour; but the proportions existing in the different states depend on the colour. This picture may be speculative and fanciful, but I can think of no other in harmony with the facts. I use the symbol P to denote the number of fibres in the green state, because I regard the sensation caused by this state to be peacock-blue rather than green.

It may easily be shown that in colour-mixing problems the above scheme leads to the same result as the Young-Helmholtz scheme. For example, suppose that light specified by R_1, P_1, B_1, Y_1 is superposed on light specified by R_2, P_2, B_2, Y_2; the mixture is specified by R_1+R_2, P_1+P_2, B_1+B_2, Y_1+Y_2; let the intensities of the components be m_1, m_2. Then

$$m_1=R_1+P_1+B_1+Y_1$$

and

$$m_2=R_2+P_2+B_2+Y_2.$$

Let

$$x_1=\frac{R_1}{R_1+P_1}, \qquad y_1=\frac{B_1}{B_1+Y_1}$$

$$x_2=\frac{R_2}{R_2+P_2}, \qquad y_2=\frac{B_2}{B_2+Y_2}.$$

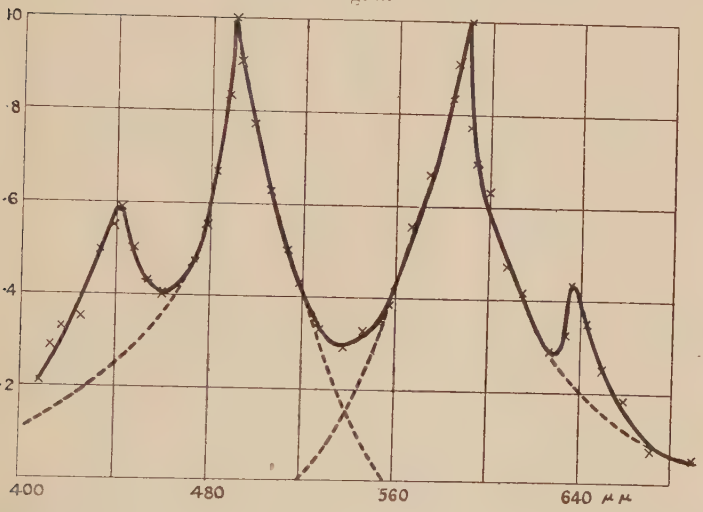
Thus, when the intensity m and the colour valencies x and y are known, the light is fully specified, and the colour can be specified by a point on a diagram with the coordinates x, y. Let the coordinates specifying the resultant colour be $\bar{x}$, $\bar{y}$. Then, by definition,

$$\bar{x} = \frac{R_1 + R_2}{R_1 + R_2 + P_1 + P_2} = \frac{2(R_1 + R_2)}{m_1 + m_2}.$$

But

$$x_1 = \frac{2R_1}{m_1}, \quad x_2 = \frac{2R_2}{m_2}.$$

Fig. 2.



Hence

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2},$$

and there is a similar equation for $\bar{y}$. The centre of gravity construction holds for the mixing of colours.

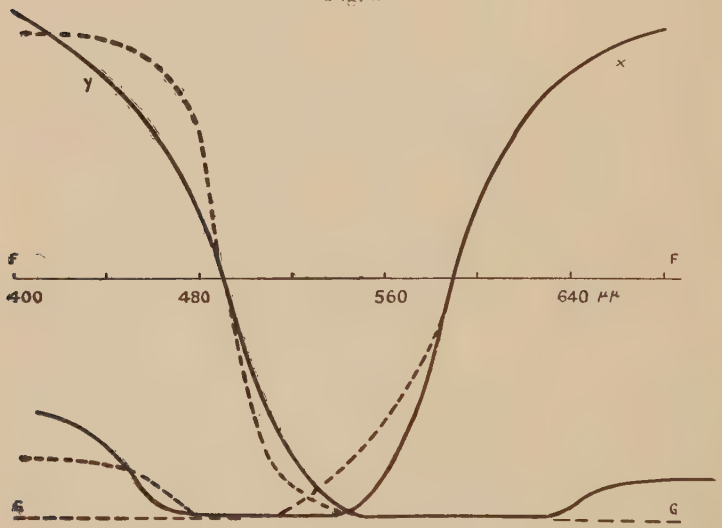
It is an immense simplification to be able to use Cartesian coordinates instead of the trilinears of the Young-Helmholtz theory.

The rate of variation of hue in the spectrum with wavelength has been investigated very carefully by O. Steindler * and L. A. Jones †. Their results are in agreement. Fig. 2

* *Sitz. d. Wien. Akad.* cxv. ii *a*, p. 115 (1906).
† Journ. Opt. Soc. Amer. p. 63 (1917).

gives the final results of L. A. Jones, slightly smoothed, the abscissæ being wave-lengths and the ordinates the reciprocal of the smallest change of wave-length that could be detected at the point in question, measured in $\mu\mu$. The curve has maxima of sensibility at yellow and peacock-blue and smaller maxima in the red and violet. The seven turning values recall Newton's seven colours of the spectrum, and it may be pointed out in passing that if an observer is asked to mark the positions of blue, peacock-blue, green, yellow, and orange in the spectrum he points exactly to five of these turning values, and if he is asked to mark the positions of red and violet he points a little to the outside of the other two.

Fig. 3



No physicist can fail to be impressed by the symmetry of the curve, especially when he knows that colour, apart from intensity, is a function of two variables. The temptation to analyse the curve into the sum of four curves is a great one. I have done so by means of the dotted lines, drawing them to make the component curves as smooth as possible. It is natural to regard these component curves as the rate of change of the colour valencies, and to associate each big maximum with a little one. In this way I have determined the colour valency curves, as shown by the full lines in fig. 3. Conversely, the curves of fig. 2 are obtained

by differentiating the curves of fig. 3. In drawing the valency curves the ordinates are indeterminate to the extent of an additive constant. The latter may be chosen so as to make the valency curves cross the axis at the principal maxima of the hue discrimination curves, or it may be chosen so as to make all the ordinates positive.

So far the values of the quantities R, P, B, and Y belonging to every wave-length have been chosen to satisfy the colour mixture data. They are, as has been mentioned, equivalent to three independent variables, and can be expressed in terms of the Young-Helmholtz primaries. Let us now tentatively assume R and B proportional to distances above FF in fig. 3 and P and Y proportional to distances below FF. Then x and y are proportional to distances above GG, the scale being chosen so as to make their maximum values unity. We consequently define the same quantities over again to satisfy the hue discrimination data, *i.e.*, in an entirely independent manner, and there is no guarantee that the two definitions will agree. But if x and y are graphed on a coordinate diagram we obtain a figure which is very like the spectrum curve on the colour diagram as determined by Maxwell and others.

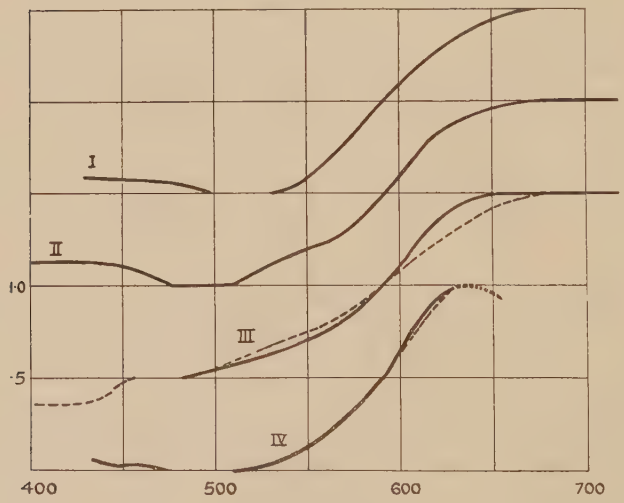
I have shown in a former paper * that the colour diagram, *i.e.*, the centre of gravity construction used in colour mixture problems, can be drawn in a fourfold infinity of ways for any three primary colours. When the positions of the first two colours are chosen we have two constants at our disposal for fixing the position of the third, and when the unit in which the first colour is measured is specified we have at our disposal the units for measuring the second and third. In the past these four constants have been chosen from a variety of considerations, to make the triangle equilateral, to have white at the centroid, to have the spectrum curve touching two of the sides, and so on. But the choice has always been made from the standpoint of the Young-Helmholtz theory. What conditions are required by the standpoint of the present paper ?

The spectrum curve on the colour diagram contains two straight parts. I think that the diagram should be adjusted so as to get these perpendicular to one another, and that their point of intersection should be taken as

* Phil. Mag. ix. p. 1130 (1930).

origin. This uses up one of our constants. Position will then be specified in Cartesian coordinates parallel to these two sides, and in no circumstances will trilinear coordinates be employed. Next the maximum x and y will be made of equal length, each unity. There is no reason why the one direction should have a preference over the other. This fixes a second constant. The third and fourth should be used to bring the principal maxima of hue discrimination halfway along the sides, *i. e.*, the coordinates of these points should be 0·5, 0 and 0, 0·5. By drawing the curve in this way we shall bring it into as close agreement with the hue discrimination data as possible.

Fig. 4.

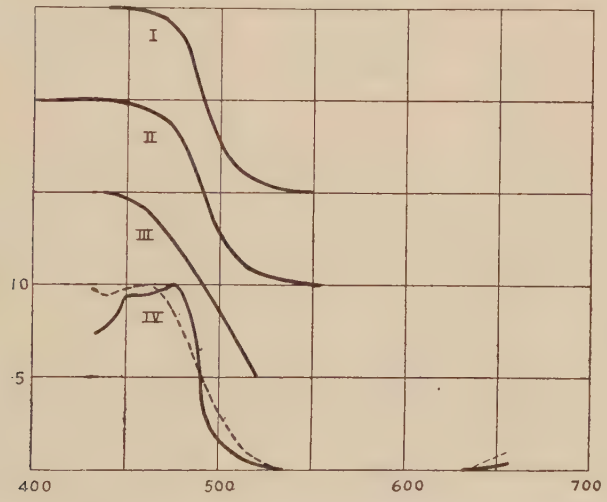


Determinations of the colours of the spectrum in terms of the primary colours have been made by Maxwell * for two observers K and J (himself), by König and Dieterici † for themselves, by Abney ‡ twice, and by Dow and myself §. I have reduced the results to a common basis in accordance with the principles of the preceding paragraph, and they are represented in figs. 4 and 5, fig. 4 giving x, the red-green valency, and fig. 5 y, the yellow-

* Phil. Trans. cl. p. 57 (1861).
† *Sitz. d. Berl. Akad.* p. 805 (1886).
‡ Phil. Trans. cxciii. A, p. 259 (1900); and ccv. A, p. 333 (1906).
§ Phil. Mag. xlv. p. 169 (1923).

blue valency. The curve no. I in each case gives the results of Dow and myself ; the readings are for his eye, verified by myself. The next curve, no. II, gives the results of König and Dieterici ; they made separate determinations, but the differences are trifling, and do not show in the diagram. No. III gives Abney's results, the 1900 determination by the full line and the 1906 determination by the broken line ; both determinations agree as regards the blue-yellow valency. No. IV gives Maxwell's own observations by the full line and those of his co-observer by the broken line. In fig. 4 I and II are in fair agreement, but it is impossible to resist the con-

Fig. 5.



clusion that Abney's 1906 determination is wrong in the violet. The deviation of Maxwell's co-observer in the red also appears to be error of observation. In fig. 5 I and II are again in fair agreement ; the two curves IV are very irregular in the blue, but they show an extremely interesting increase in the yellow-blue valency in the red.

I have entered curves II of figs. 4 and 5 in fig. 3 by a broken line. Thus this fig. represents the valencies determined in two independent manners, from the hue discrimination data and from the colour mixture data. If I had used the results obtained by Dow and myself instead of the results of König and Dieterici, the agreement with

the full line would on the whole have been just about as good. The chief difference is in y in the red; here the hue discrimination curve shows an increase found neither by König, Dieterici, Dow, nor myself, but given by Maxwell. The late Lord Rayleigh * investigated this question, as to whether the red end of the spectrum became slightly blue or not, and decided that it had no tendency towards blue; naturally the greatest weight must be attached to his decision, but nevertheless the question seems to deserve reinvestigation.

The curves brought into contrast in fig. 3 are by different observers, L. A. Jones on the one hand, and König and Dieterici on the other. Much of the difference is undoubtedly due to a difference in the colour vision of the observers. There are, however, two other possible causes. In resolving fig. 2 into separate curves I assumed that the rate of change of hue with wave-length was given by

$$\frac{dx}{d\lambda}+\frac{dy}{d\lambda}, \quad \text{not by} \quad \sqrt{\left(\frac{dx}{d\lambda}\right)^2+\left(\frac{dy}{d\lambda}\right)^2}.$$

The second expression has more theoretical justification; it would lead to approximately the same numerical results. Again I am doubtful whether rate of change of hue with wave-length should follow $\frac{dx}{d\lambda}$ and $\frac{dy}{d\lambda}$ closely. These expressions should undoubtedly determine it approximately, but the result is influenced by practice, memory, association of ideas, etc. The results of Jones and Lowry † on mixtures of white and spectral colours show that the eye is much better at distinguishing changes in saturation in the neighbourhood of white and of complete saturation than in the middle of the range; red, for example, and white are definite mental pictures, and the eye uses them as points of reference from which to make its judgments.

In the explanation of colour vision put forward in the present paper it is assumed that the cones have different intensity thresholds and that all the cones are sensitive to all the colours of the spectrum. This necessitates modifying one of the statements of a former paper. The relative

* 'Scientific Papers,' v. p. 569.

† Journ. Opt. Soc. Amer. xiii. p. 25 (1926).

visibility of the different colours of the spectrum is given by

$$e^{-\frac{(\log\lambda-\log\lambda_0)^2}{2\sigma_1^2}}.$$

I assumed in the former paper that this expression applied to the cones; I now transfer it to the photochemical substance. In other words, if two different monochromatic radiations fall upon the photochemical substance liberating streams of electrons, this expression gives the ratio of the energy of the stream of electrons to the energy of the incident radiation. The energy of the stream of electrons is strictly proportional to the intensity of the light measured in metre-candles. When the electrons fall upon the cones the number of fibres excited is proportional to the intensity of the sensation, in accordance with the expression

$$S = \int_{-\infty}^{\log I} e^{-\frac{(\log I-\log I_0)^2}{2\sigma^2}}\, d(\log I).$$

In this expression I must be expressed in metre-candles.

The question is naturally asked as to how the theory now put forward fares with regard to the explanation of after-images, simultaneous contrast, and the growth and decay of the sensation. It is impossible as yet to bring anything decisive from this region either for or against it. We can form a clear picture only of part of the process, namely, of the action of light on the photochemical substance; it is impossible to follow quantitatively the stimulation of the cones or the action of the synapse. So we are only able to state how far this part of the process goes towards a complete explanation, and leave what is outstanding to the other links in the chain.

The action of the flicker photometer is based on the fact that colour flicker disappears at a much lower frequency than intensity flicker. It is thus natural to conclude that they are due to separate processes, and H. E. Ives, who has done much of the fundamental work on this photometer, has already put forward this conclusion in a tentative manner. The photochemical substance undoubtedly plays a large role in determining the rate of growth and decay of the sensation. If a steady light falls on it the rate of emission of electrons is increased to a maximum, then diminishes as the supply of substance is exhausted, and finally becomes constant when the rate of supply of fresh

substance balances its rate of destruction. Hecht has put forward reasons for the view, that fresh substance is supplied by the products of decomposition combining together according to the equation for a bimolecular reaction.

Now the experiments of Broca and Sulzer show that, when a steady light falls upon the retina, the sensation at first overshoots its final value in precisely this way, and Adrian and Matthews found a similar variation in the action currents obtained from the optic nerve of the conger eel. But according to Broca and Sulzer the curve of response varies with the wave-length of the light, the maximum being very pronounced in the case of the blue. This variation cannot be explained by the photochemical substance.

If the incident light is cut off the supply of electrons at once ceases, but those already emitted and on their road towards the cones have still to act on the latter. Thus the positive after image is explained simply by the inertia of the process. If now fresh light falls on the whole retina the supply of photochemical substance is much less where light has previously acted. Thus we have the explanation of the negative after image.

Again we have assumed that each molecule of the photochemical substance is susceptible only to a narrow range of wave-lengths. Thus, if red light is incident, the supply of molecules susceptible to the complementary colour blue-green is unaffected, and the negative after image appears in this complementary colour. The high degree of saturation of a colour falling on the after image of its complementary can be explained by the exhaustion of the molecules spreading in from the regions of the spectrum on both sides, and thus narrowing the band on which it can act. "Light chaos" or "light dust" is possibly due to spontaneous decomposition of the photochemical substance. The judgment of brightness will be affected at the edge of an image, since the surrounding region is used as a level of reference. Thus the mental factor comes in. These points all deserve careful mathematical investigation, but the data should be obtained from groups of observers; individuals vary very much with regard to these secondary effects. Hecht has devoted much attention to the properties of the photochemical substance, but he uses its time variation to explain dark adaptation. For reasons

mentioned in a former paper I cannot accept this view.

There is one well known feature of after images which cannot be explained by the properties of a photochemical substance, namely, their periodic recurrence. If the eye is fixed on a bright green disk in a dark room for thirty seconds, and the disk is replaced by a white screen, a magenta after image appears and reappears on the screen at intervals of three or four seconds. The effect is very striking when the white screen fills the full aperture of the eye. I am inclined to ascribe it to some sort of oscillation set up in the synaptic layer in the retina. In their experiments on the eye of the conger eel Adrian and Matthews * found, that when the whole retina was subjected to a steady uniform illumination, the action current sometimes showed a rhythmic succession of large waves varying perhaps from 25 to 6 per second. They established the fact that this discharge was due to a large number of neurones becoming linked together and working in unison. Now steady uniform illumination is a necessary condition for the recurring after image, and it possibly puts the retina into a state favourable to the production of oscillations. However, this is very speculative.

If red light falls on a region of the retina, the molecules of the photochemical substance sensitive to red become exhausted in this region ; if we can assume an equilibrium between these molecules as a class, then some will diffuse in from the neighbouring area on account of the balance being disturbed. This appears to be the only way of explaining simultaneous contrast by the photochemical substance ; I regard it as a very improbable explanation.

Note on the Relative Visibility of the Different Colours of the Spectrum.

It was stated in a former paper †, that the relative visibility of the different colours of the spectrum is a probability curve, when graphed against log λ as abscissa, and a diagram was given showing how much the values given by Gibson and Tyndall differed from the probability curve. I desire to go into the matter more fully here, as I have now data that were wanting when the former paper was written.

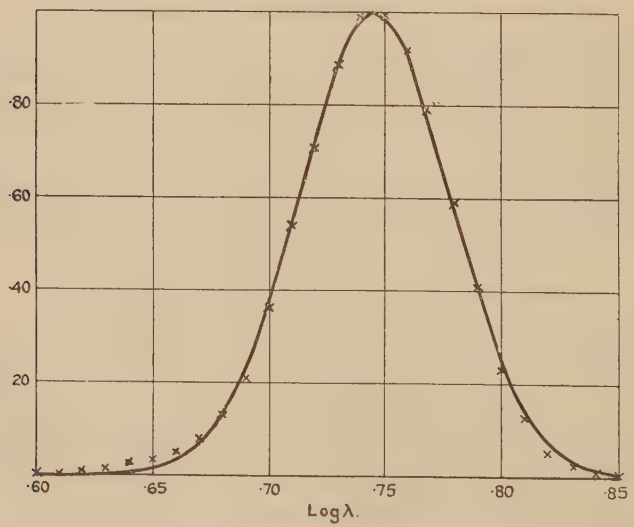
* Journ of Physiology, lxv. p. 273 (1928).
† Phil. Mag. x. p. 416 (1930).

Values of the visibility have been recommended by Ives, Priest, the I.E.S., and Gibson and Tyndall. I assumed that these are represented by the probability curve, and calculated the mean and the standard deviation σ by the method of moments. I then adjusted the height of the theoretical curve to make the agreement as good as possible, and finally calculated the root mean square of the deviations between the experimental and theoretical values for the range 400–700 $\mu\mu$. The following table gives the results :—

TABLE I.

	Mean log λ.	σ.	Max. of theoretical curve.	Root mean square of deviations.
Ives	·7456	·03292	1·00	·0152
Priest	·7453	·03286	1·023	·0219
I. E. S.	·7455	·03173	1·024	·0196
Gibson and Tyndall.	·7451	·03300	1·024	·0237

Fig. 6.



Thus Ives' values give the best agreement, and in their case no vertical adjustment was necessary. In fig. 6 the

crosses represent Ives' values and the curve the probability curve with the above constants. The following table gives the theoretical and experimental values throughout the spectrum. :—

TABLE II.

Visibility Curve recommended by Ives.

Log λ.	Experimental.	Theoretical.	Difference.
·60	·0024	—	+·0024
·61	·0030	·00025	·0027
·62	·006	·00075	·0052
·63	·014	·0023	·012
·64	·024	·0057	·018
·65	·036	·0185	·0175
·66	·051	·0390	·012
·67	·079	·0707	·008
·68	·130	·135	−·005
·69	·211	·239	−·028
·70	·360	·379	−·019
·71	·540	·555	−·015
·72	·710	·735	−·025
·73	·885	·891	−·006
·74	·988	·983	+·005
·75	·990	·989	·001
·76	·920	·906	·014
·77	·790	·757	·033
·78	·596	·579	·017
·79	·406	·401	·005
·80	·232	·251	−·019
·81	·126	·146	−·020
·82	·055	·077	−·022
·83	·027	·037	−·010
·84	·012	·016	−·004
·85	·004	·006	−·002

The visibility curve has been measured by A. König * for a number of different intensity levels, and his results have been reduced by Nutting †. I have graphed Nutting's figures against log λ; they agree with a probability curve within experimental error, and this curve shifts towards the blue end of the spectrum as the intensity is reduced, in accordance with the Purkinje effect. I have calculated the mean and the standard deviation σ for each of the different intensity levels, and the results are given in the following table :—

Table III.

	Log intensity.	Log mean, λ.	σ.	Log I.U.
S.....	$\bar{4}\cdot4$?	·702	·0290	$\bar{4}\cdot0$
A.....	$\bar{3}\cdot4$	·701	·0290	$\bar{3}\cdot0$
B.....	$\bar{2}\cdot6$	·702	·0276	$\bar{2}\cdot2$
C.....	1·8	·704	·0310	$\bar{1}\cdot4$
D.....	0·4	·711	·0344	0·0
E.....	1·0	·729	·0340	0·6
F.....	1·6	·734	·0361	1·2
G.....	2·2	·737	·0373	1·8
H.....	2·8	·740	·0357	2·4

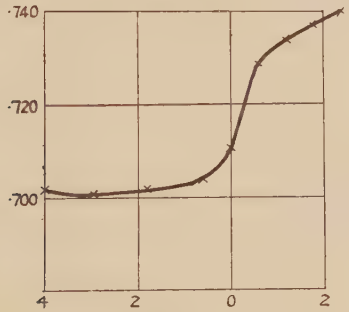
The letter is König's name for the intensity in question, and the second column gives the logarithm of the intensity expressed in König's arbitrary unit. According to his definition this unit is one-fifth of the photon, the unit introduced by Troland, or two of the illumination units introduced by Ives. In the previous papers of this series I have expressed results in photons, but in future intend to use the illumination unit of Ives. It is the degree of illumination of the image on the retina, when the eye views normally a magnesium oxide surface illuminated normally by one metre-candle, and has the advantage that the definition does not involve the perfectly diffusing surface nor the lambert, a unit that has never established itself.

* 'Collected Works,' p. 144.
† Bull. Bur. Stds. vii. p. 238 (1911).

As mentioned in a previous paper, I have found that the value ascribed by König to his unit is wrong. To bring its log to log I.U. (illumination units) we must subtract 0·4. In the previous paper the value 0·2 was given, but in view of more recent work 0·4 seems better. Thus König's intensity unit is about two-fifths of the I.U. The last column gives the intensity levels in log I.U. on this basis. Fig. 7 shows the shift of the maximum with the change in intensity; dark adaptation is apparently complete at about 0·04 I.U.

Hecht and Williams * found that the scotopic visibility curve had a maximum at log $\lambda = \cdot 705$. Ives' curve has its maximum at ·7456. These values agree quite well with

Fig. 7.



König's end values. The width of Ives' curve is given by $\sigma = \cdot 03292$, and presumably the width of Hecht and Williams's curve is given by the same number. The mean of König's figures ·0325 happens to be almost exactly the same.

In a former paper I regarded the Purkinje shift as due to one stationary curve decreasing and another one growing up alongside of it. I now adopt the view that the same curve moves bodily across from the one position to the other. In the same paper, in referring to the work of Hecht and Williams I omitted the name of Mr. R. E. Williams. I desire here to correct this omission.

* Journ. Gen. Physiol. v. p. 1 (1922).

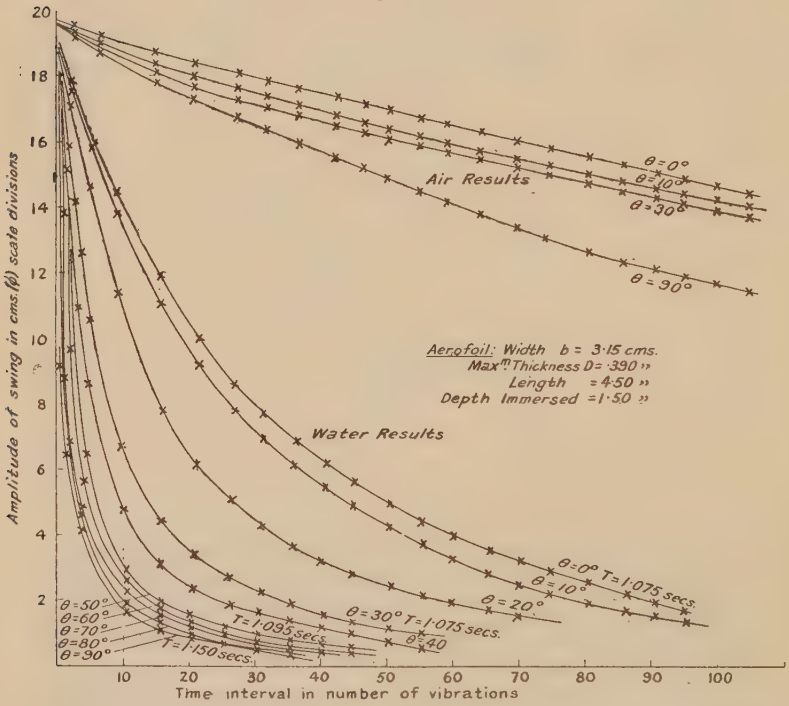
XLV. *The Free Damping of an Aerofoil Pendulum immersed in a Fluid.*

By E. Tyler, *D.Sc., F.Inst.P., Lecturer, Physics Department, College of Technology, Leicester* *.

Introduction.

WHEN an aerofoil section is attached to the lower end of a vertical reed and is free to vibrate with its lower end in a stream of fluid at right angles to the direction

Fig. 1.



of flow, the periodic forces applied to it, as a consequence of the vortex formation produced in its wake, will depend on the strength of the vortices, apart from the condition of resonance existing between these forces and the vortex formation. At low angles of incidence the strength of the vortices formed is small, whereas the damping effect on any vibration imparted to the aerofoil pendulum is much greater than at larger angles of incidence.

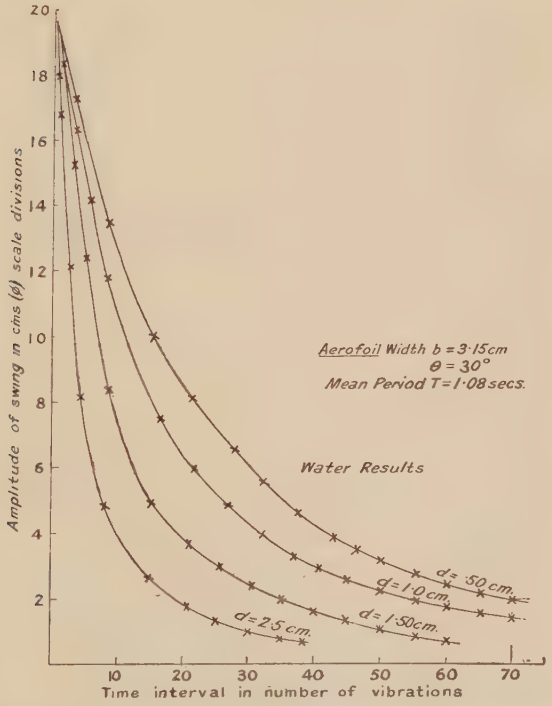
* Communicated by the Author.

It was therefore thought that it would be interesting to investigate how the free vibrations of such a pendulum in a stationary fluid were affected when the aerofoil section was inclined at varying angles to the plane of vibration.

Experimental Arrangement.

A long steel blade was clamped at its upper end, and attached to the lower end was a pointer and a small screw,

Fig. 2.



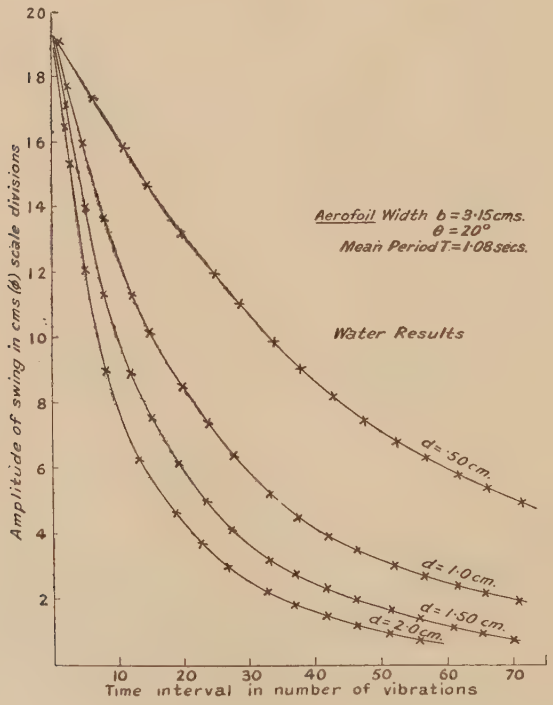
fixed into a clamp carrying the aerofoil section and a second pointer. This clamp was capable of rotating in a horizontal plane, and thus the aerofoil attached to it could be set at any required angle. By fixing a mirror to the lower end of the blade and using a lamp and scale, the motion of the pendulum could be followed very clearly.

The problem was attacked from two directions. Firstly, the aerofoil was immersed to a fixed depth in a large tank of water, then displaced about 10° from its mean position of

rest, and amplitudes of swing observed on the scale after intervals of two vibrations, until the motion had become practically damped out. The same amplitude was reached each time before observations were made at successive intervals.

This procedure was repeated for varying angles of inclination made by the flat undersurface of the aerofoil with the plane of vibration.

Fig. 3.



In the second investigation the aerofoil section was set at a known inclination to the plane of vibration, and observations made on the amplitude of swing as before, always commencing readings at the same amplitude. This process was repeated for different depths of immersion.

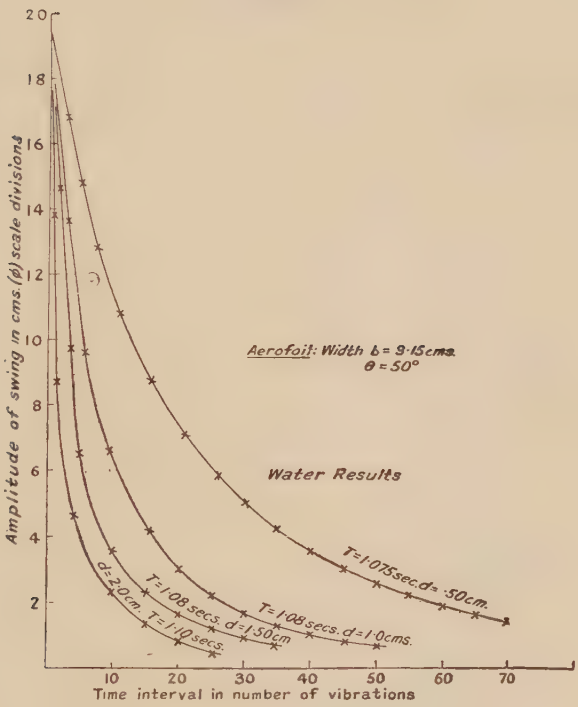
Fig. 1 shows the decay curves at different angles of inclination for a constant depth of immersion $d = 1{\cdot}50$ cm., and figs. 2, 3, 4. and 5, the effect of lowering the aerofoil to different depths.

Theory.

The equation of motion of a vibrating reed loaded with an aerofoil section partly immersed in a fluid may be written as

$$\frac{d^2\phi}{dt^2} + k\frac{d\phi}{dt} + n^2\phi = 0,$$

Fig. 4.



where ϕ = the angular displacement at any instant, and k = a constant dependent upon the frictional resistance of the system, which consists of two components : (1) that due to the reed and screw attachment together with the part of the aerofoil not immersed, say k_1, and (2) that experienced by the immersed part of the aerofoil, say k_θ.

Thus, at a given angle of inclination of the aerofoil, $\theta°$, to the plane of vibration, the equation of motion is given as

$$\frac{d^2\phi}{dt^2} + (k_\theta + k_1)\frac{d\phi}{dt} + n^2\phi = 0,$$

the solution of which is

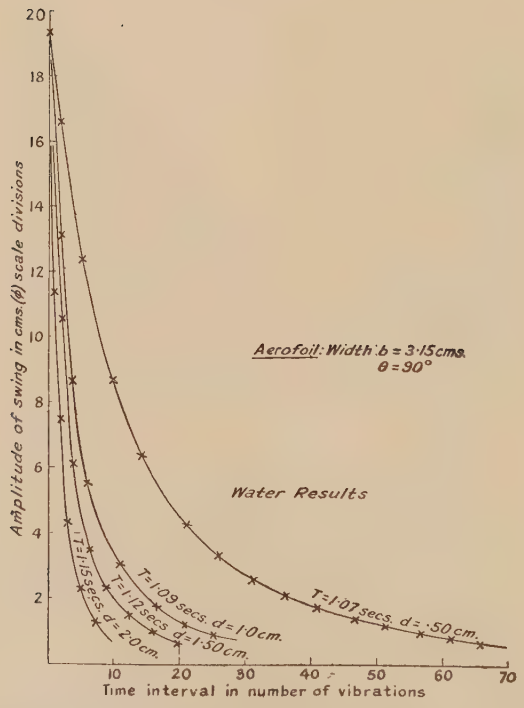
$$\phi = Ae^{-\frac{1}{2}(k_\theta + k_1).t} \cos \sqrt{n^2 - \tfrac{1}{4}(k_\theta + k_1)^2}\,.\,t,$$

and for successive amplitudes of swing

$$\frac{\phi_0}{\phi_1} = \frac{\phi_1}{\phi_2} = \frac{\phi_2}{\phi_3} = \text{etc.} = e^{\frac{1}{2}\frac{(k_\theta + k_1)\pi}{\sqrt{n^2 - \frac{1}{4}(k_\theta + k_1)^2}}} = \delta,$$

$$\therefore \quad \log_e \delta = \frac{1}{2}\frac{(k_\theta + k_1)\pi}{\sqrt{n^2 - \frac{1}{4}(k_\theta + k_1)^2}} = \lambda_\theta,$$

Fig. 5.



where λ_θ = the log dec of the system at inclination of aerofoil $\theta°$ to the plane of vibration.

If T_θ = the period of vibration of the system at $\theta°$, then

$$\lambda_\theta = \tfrac{1}{4}(k_\theta + k_1)T_\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, if λ_0 = the log dec of the system when $\theta = 0$, then

$$k_\theta = k_0, \quad \text{and} \quad \lambda_0 = \tfrac{1}{4}(k_0 + k_1)T_0. \quad . \quad . \quad . \quad (2)$$

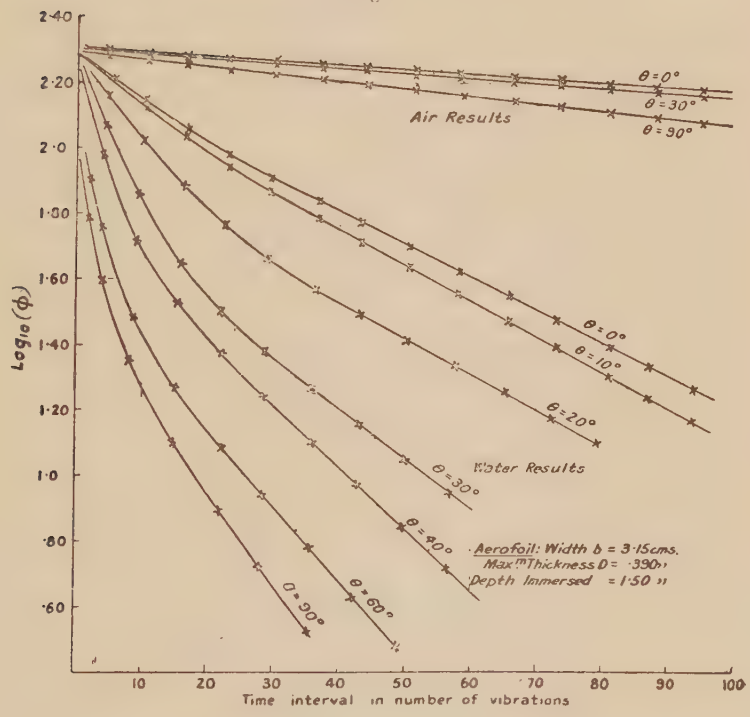
From equations (1) and (2) we have

$$\lambda_\theta - \lambda_0 = \tfrac{1}{4}\{(k_\theta + k_1)T_\theta - (k_0 + k_1)T_0\}$$
$$= \tfrac{1}{4}\{k_\theta T_\theta - k_0 T_0 + k_1(T_\theta - T_0)\}.$$

Since $T_\theta \doteq T_0$ and k_1 is small in the experiments,

$$\lambda_\theta - \lambda_0 \doteq \tfrac{1}{4} T_\theta (k_\theta - k_0). \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Fig. 6.



Assuming now, that the difference in the damping coefficients is proportional to the effective area of the immersed part of the aerofoil, $k_\theta - k_0$ will be proportional to the change in effective area between $\theta = 0$ and $\theta°$, *i. e.*,

$$k_\theta - k_0 \propto d(b \,.\, \sin\theta - D),$$

or

$$k_\theta - k_0 = K \,.\, d(b \,.\, \sin\theta - D),$$

whence

$$\lambda_\theta - \lambda_0 = K_1 T_\theta d \,.\, (b \,.\, \sin\theta - D), \quad . \quad . \quad . \quad (4)$$

where K and K_1 are constants,

d = the depth of the immersed part,

and D = the maximum thickness of the aerofoil, and the effective width at $\theta=0$.

Thus at a constant depth, assuming that there is very little variation of T_θ for angles θ_1 and θ_2, we get

$$\frac{k_{\theta_1}-k_0}{k_{\theta_2}-k_0}=\frac{b\,.\sin\theta_1-D}{b\,.\sin\theta_2-D},$$

and since $k_\theta-k_0 \propto \lambda_\theta-\lambda_0$ (see equation (3)) ; therefore

$$\frac{\lambda_{\theta_1}-\lambda_0}{\lambda_{\theta_2}-\lambda_0}=\frac{b\,.\sin\theta_1-D}{b\,.\sin\theta_2-D}. \quad . \quad . \quad . \quad . \quad (5)$$

Equation (5) is therefore representative of the conditions existing when the damping effect is not too large so as to alter the natural period of the system, and the values of λ_θ will only apply to the linear parts of the log (ϕ)—time graph in fig. 6. This has been done in the following examples.

Case 1 (*a*).—*Aerofoil in water at constant depth.*

$d = 1{\cdot}50$ cm., $b = 3{\cdot}15$ cm., $D = {\cdot}39$ cm.,

length = 4·50 cm., mass = 32 gm.

when

$\theta_0 = 0°$, $T_0 = 1{\cdot}075$ sec., $\frac{\lambda_0}{T_0} = {\cdot}02170$,

$\theta_1 = 30°$, $T_{30} = 1{\cdot}075$ sec., $\frac{\lambda_{30}}{T_{30}} = {\cdot}03452$,

$\theta_2 = 60°$, $T_{60} = 1{\cdot}100$ sec., $\frac{\lambda_{60}}{T_{60}} = {\cdot}04760$;

thus giving $$\frac{\lambda_{\theta_2}-\lambda_0}{\lambda_{\theta_1}-\lambda_0}=\frac{{\cdot}02590}{{\cdot}01308}=1{\cdot}98.$$

Substituting the values of b and D in the expression

$$\frac{b\,.\sin\theta_2-D}{b\,.\sin\theta_1-D},$$

we get $$\frac{3{\cdot}15\times{\cdot}866-{\cdot}39}{1{\cdot}575-{\cdot}39}=\frac{2{\cdot}340}{1{\cdot}185}=1{\cdot}98,$$

which is in excellent agreement with the former value 1·98.

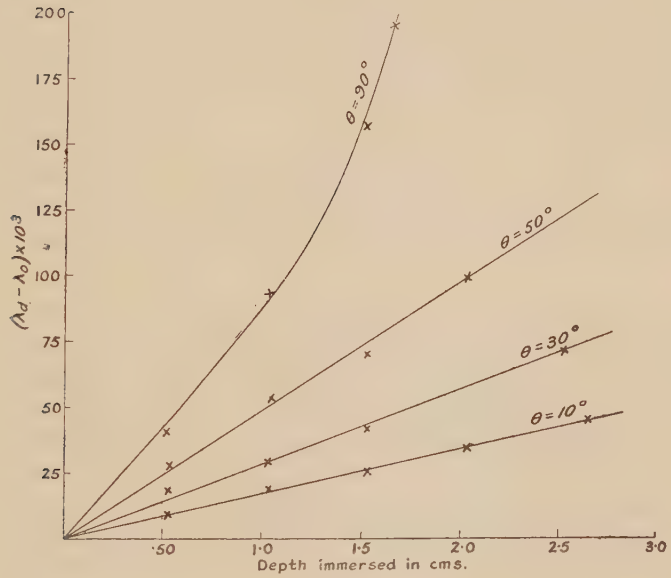
Case 1 (*b*).

$$\theta_0 = 0°, \quad T_0 = 1{\cdot}075 \text{ sec.}, \quad \frac{\lambda_0}{T_0} = {\cdot}02170,$$

$$\theta_1 = 30°, \quad T_{30} = 1{\cdot}075 \text{ sec.}, \quad \frac{\lambda_{30}}{T_{30}} = {\cdot}03452,$$

$$\theta_2 = 90°, \quad T_{90} = 1{\cdot}150 \text{ sec.}, \quad \frac{\lambda_{90}}{T_{90}} = {\cdot}05600\ ;$$

Fig. 7.



thus giving $$\frac{\lambda_{\theta_2} - \lambda_0}{\lambda_{\theta_1} - \lambda_0} = \frac{{\cdot}03450}{{\cdot}01308} = 2{\cdot}63,$$

whereas

$$\frac{b \,.\, \sin\theta_2 - D}{b \,.\, \sin\theta_1 - D} \text{ gives } \frac{3{\cdot}15 - {\cdot}39}{1{\cdot}575 - {\cdot}39} = \frac{2{\cdot}76}{1{\cdot}185} = 2{\cdot}34.$$

The discrepancy between these latter results may be attributed to the damping effect being more appreciable ($\theta° = 90$) to alter the period of the system, causing a departure from the simple difference ratio relation, and whenever there is excessive damping (*i. e.* λ_θ is large) the simple relation no longer holds.

Case 2.—*Effect of depth on log dec at constant inclination.*

Referring to equation (4), if we consider θ constant and put λ_d and λ_0 for the log decs at depth d and 0 cm., and again supposing T_θ is practically constant, then

$$\lambda_d - \lambda_0 = K_2 d,$$

where K_2 is a constant for a given value of θ.

This is a linear equation, and a plot of $\lambda_d - \lambda_0$ against d establishes this equality. Fig. 7 shows such results plotted, and it will be observed that for each setting of the aerofoil, $\lambda_d - \lambda_0 \propto d$, the slopes of the graphs increase with increasing values of θ. It is only when $\lambda_d - \lambda_0$ is greater than ·10 that there is a departure from this law, exhibited in the extreme case of $\theta = 90°$ and $d = 2{\cdot}0$ cm., when under such excessive damping

$$\left(\frac{\lambda_2 - \lambda_0}{\lambda_1 - \lambda_0}\right) = \left(\frac{d_2}{d_1}\right)^2,$$

where λ_1 and λ_2 are the log decs for depths d_1 and d_2 cm.

Summary.

The effect on the free damping of an aerofoil pendulum immersed in a stationary fluid (water) at varying depths, and set at different angles of inclination $\theta°$ to the plane of vibration is investigated. The results show that the natural period of vibration of the pendulum as determined in air is very little affected by immersing the pendulum in the fluid. If λ_0, λ_{θ_1}, and λ_{θ_2} are the log decs at $\theta = 0$, $\theta_1°$ and $\theta_2°$ at constant depth of immersion d cm., and b and D = the width and maximum thickness of the aerofoil respectively, then

$$\frac{\lambda_{\theta_2} - \lambda_0}{\lambda_{\theta_1} - \lambda_0} = \frac{b \sin \theta_2 - D}{b \sin \theta_1 - D}.$$

Furthermore, if λ_0, λ_1, λ_2, and λ_d are the log decs at a given inclination θ, for depths of immersion 0, d_1, d_2, and d cm.,

$$\lambda_d - \lambda_0 = Kd \text{ where } \lambda_d < {\cdot}10 \text{ (K = a constant)}$$

and $\dfrac{\lambda_2 - \lambda_0}{\lambda_1 - \lambda_0} = \left(\dfrac{d_2}{d_1}\right)^2$ when $\lambda_d > {\cdot}10$ for excessive damping.

The effect in other liquids is under further investigation.

XLVI. *The Electrode Potential and the Solvent: the Solvation Activity Coefficient.* *By* Frederick Karl Victor Koch, *Ph.D.*, *Beit Scientific Research Fellow* *.

§ 1. *Thermodynamic Considerations.*

THE free energy of transference (A) of an ion from a vacuum to a solution of the ion of specified concentration in a given solvent may be split up into two parts : (1) the free energy of transference (A_s) of the ion from a vacuum to the pure solvent, *i. e.*, to an infinitely dilute solution of the ion in that solvent, and (2) the free energy of transference (A_c) of the ion from the pure solvent to a solution of specified concentration. Thus

$$A = A_s + A_c.$$

These free energies may be expressed in terms of activities :

$$A = RT \,.\, \log_e a,$$

$$A_s = RT \,.\, \log_e a_s,$$

$$A_c = RT \,.\, \log_e a_c,$$

whence

$$a = a_s \,.\, a_c,$$

where a is the activity corresponding to the complete transference, a_c is the "activity" of Lewis and Randall ('Thermodynamics,' 1923), and a_s is a quantity depending only on the nature of the ion and the solvent, and may therefore be called the *solvation activity*.

The transference of the ion from a vacuum to a solution of specified concentration in a given solvent may also be carried out *via* the element electrode. The total free energy change (A) may then be split up into (1) the free energy of transference (A_E) of the ion from a vacuum to the element electrode, and (2) the free energy of transference (A_P) of the ion from the element electrode to the solution of specified concentration. Thus

$$A = A_E + A_P.$$

These free energies may likewise be expressed in terms of activities.

$$A_E = RT \,.\, \log_e a_E,$$

$$A_P = RT \,.\, \log_e a_P,$$

whence

$$a = a_E \,.\, a_P,$$

* Communicated by Prof. James C. Philip, F.R.S.

where a_E may be called the *element activity* and a_P is a quantity which will be shown in section 2 to be identical with the Nernst solution tension, when $a_c=1$.

Now $A_P=n\,.\,F\,.\,E$, where E is the electrode potential. Hence

$$A_P=n\,.\,F\,.\,E=A-A_E=A_S+A_C-A_E$$
$$=A_S-A_E+RT\,.\,\log_e a_C,$$

or
$$E=A_S/(n\,.\,F)-A_E/(n\,.\,F)+(RT/n\,.\,F)\,.\,\log_e a_C$$
$$=E_S-E_E+(RT/n\,.\,F)\,.\,\log_e a_C,$$

where $A_S/(n\,.\,F)=E_S$ and $A_E/(n\,.\,F)=E_E$ may be called the *solvation potential* and the *element potential* respectively. The solvation potential is the difference in potential between an ion in a vacuum and in the pure solvent, and the element potential is that between an ion in a vacuum and in the element electrode.

The absolute normal potential (E_N) is defined as the electrode potential when $a_C=1$, and therefore

$$E_N=E_S—E_E.$$

Hence it is seen that the absolute normal potential may be regarded as the *difference* of two potentials, one dependent only on the solvent and the ion and the other dependent only on the element electrode and the ion.

§ 2. *Meaning of the Nernst "Electrolytic Solution Tension."*

If the activity (a_c) be considered as proportional to the osmotic pressure (p), then the original Nernst formula

$$E=(RT/nF)\,.\,\log_e P/p$$

may be written as

$$E=(RT/nF)\,.\,\log_e P_c/a_c,$$

where P_c is the solution tension expressed in terms of activity. When $a_c=1$,

$$A_P=RT\,.\,\log_e a_P=n\,.\,F\,.\,E_N=RT\,.\,\log_e P_c,$$

and

$$a_P=a_S/a_E,$$

whence

$$P_c=a_P=a_S/a_E.$$

Since the solution tension is seen to be the *ratio* of two activities, one dependent only on the solvent and the ion

and the other dependent only on the element electrode and the ion, it may be regarded as the distribution coefficient of the ion between the pure solvent and the element electrode. The ratio of the solution tensions of a given metal in two solvents is (since a_E cancels out) the ratio of two solvation activities, *i. e.*, the distribution coefficient of the metal ion between the two solvents (*cf.* the "specific ionic distribution coefficient" of Nernst, *Z. physikal. Chem.* ii. p. 132 (1892)).

§ 3. *Values of the Normal Potential, Solvation Potential, and Element Potential.*

In the following table the values of E_S have been calculated for a number of ions in water from the free energies

Ion.	E_N (absolute).	E_S.	E_E.
Cations :			
H	+0·28	+11·4	+11·1
Li	+3·24	+ 4·8	+ 1·6
Na	+2·99	+ 4·5	+ 1·5
K	+3·20	+ 3·6	+ 0·4
Rb	+3·20	+ 3·2	(± 0·0)
Cs	—	+ 3·2	—
Tl	+0·62	+ 3·6	+ 3·0
Ca	+2·55*	+ 7·5	+ 5·0
Anions :			
Cl	—1·08	— 3·4	— 2·3
Br	—0·79	— 3·0	— 2·2
I	—0·12	— 2·5	— 2·4

* Michaelis, 'Wasserstoffionenkonzentration,' p. 144 (1922).

of hydration estimated by Fajans (*Ber. d. Phys. Ges.* xxi. p. 549 (1919); *Naturwissenschaften*, ix. p. 729 (1921); see also correction by Born, *Z. Physik*, i. p. 45 (1920)) on the basis of experimental data. The normal potentials are those given by Lewis and Randall (*loc. cit.* p. 433), and have been converted to the absolute scale by taking the absolute potential of hydrogen in water as +0·28 volt. E_E has been calculated from the differences (E_S—E_N). The potentials are in volts.

§ 4. *The Solvation Activity Coefficient.*

For many purposes it is convenient to express the solvation activity of an ion in a given solvent with reference to the corresponding solvation activity in water a_w (*i. e.*, the hydration activity). $a_s/a_w=\gamma_S$ may therefore be called the *solvation activity coefficient*. This procedure amounts to taking (arbitrarily) the hydration activities of all ions as unity.

The solvation activity coefficient is the factor by which the activity of S-solvated ions must be multiplied in order to give the electromotively equivalent activity of hydrated ions. The solvation activity coefficient is identical with the ionic distribution coefficient of the ion between two solvents when one solvent is water, and may be determined by a method described by the author (Koch, J. Chem. Soc. p. 269 (1928)) or by other methods (see Bjerrum and Larsson, *Z. physikal Chem.* cxxvii. p. 368 (1927)). Some values of solvation activity coefficients are given below.

Solvent : Ethyl Alcohol (Bjerrum and Larsson, *loc. cit.*).

Cations.	γ_S.	Anions.	γ_S.
H	$3{\cdot}2\times10^{2}$	Cl	$3{\cdot}2\times10^{2}$
Li	$6{\cdot}3\times10^{2}$	Br	63
Na	$6{\cdot}3\times10^{3}$	I.....	25
K	$7{\cdot}9\times10^{3}$	ClO_4	5·0
Cs.....	$1{\cdot}0\times10^{4}$	Benzoate	20
Ag.....	$1{\cdot}3\times10^{2}$	O-nitrobenzoate ..	50
NH_4	$5{\cdot}0\times10^{2}$	M-nitrobenzoate ..	5·0
$N(CH_3)_4$	$1{\cdot}0\times10^{3}$	P-nitrobenzoate ..	6·3
$N(C_2H_5)_4$	$5{\cdot}0\times10^{2}$	Salicylate	16
		Laurate	4×10^{-3}
		Myristate	$3{\cdot}2\times10^{-4}$
		Palmitate	$2{\cdot}0\times10^{-4}$
		Stearate	$1{\cdot}0\times10^{-4}$

Let the activities of A-solvated, B-solvated, C-solvated ions in a mixed solvent (having constituents A, B, C) be a_A, a_B, a_C, and let the respective solvation activity coefficients of the ions be γ_A, γ_B, γ_C. Then, on the *assumption* that the respective degrees of solvation of the ions in the mixture are the *same* as in the pure solvents,

$$E=E_W-E_E-(RT/nF)\cdot\log_e(\gamma_A a_A+\gamma_B a_B+\gamma_C a_C),$$

where E is the potential difference between the electrode and the solution, E_E is the element potential for that electrode, and E_W is the potential difference which would exist between the electrode and the solution if all the ions were hydrated. This equation has been applied by the author (J. Chem. Soc. 1930, p. 2054, equation 2) to the investigation of complex ions in solution.

§ 5. *Electrode Potential and Dielectric Constant.*

(a) *The Born Formula.*—Born (*loc. cit.*) has shown that the free energy of solvation of an ion may be simply

The Silver Ion (Koch, J. Chem. Soc. p. 269 (1928)).

Solvent.	Dielectric constant.	γ_S.
Pyridine	12·4	$2{\cdot}1\times10^{-6}$
Aniline	6·9	$4{\cdot}9\times10^{-3}$
Acetonitrile	36·4	$5{\cdot}2\times10^{-2}$
Propionitrile	27·5	0·76
Phenylacetonitrile	15	8·6
Ethyl cyanoacetate	27·7	8·8
Benzonitrile	26·3	10·9
Methyl alcohol	35·4	81
Ethyl alcohol	25·4	150
Acetone	21	6040

γ_S (Koch, J. Chem. Soc. p. 1551 (1930)).

Ion.	Methyl alcohol.	Ethyl alcohol.
Cl	13·2	123
Br	8·6	54
I	3·0	15

calculated on the assumptions (1) that the ion is a *rigid sphere*, and (2) that it is transferred from a vacuum to a *continuous* dielectric medium. Thus

$$A_S(\text{Born}) = 1{\cdot}194\times10^{-11}\,.\,N\,.\,e^2\,.\,z^2(1-1/\epsilon)/r_i \text{ kg. cal.},$$

where ϵ is the dielectric constant of the medium, e is the electronic charge, N is the Avagadro number, z is the valency, and r_i is the radius of the ion.

A theoretical solvation activity and solvation activity coefficient can, of course, be calculated from A_S (Born).

The two conditions of the validity of Born's formula are tantamount to the condition that there should be no specific ionic-molecular interaction. When this condition is fulfilled the electrode potential is given by a formula of the type $E = a + b/\varepsilon$ (Brodsky's formula, *Z. physikal. Chem.* cxxi. p. 1 (1926)), where a and b are constants characteristic of the ion and independent of the solvent. This formula has been shown to hold approximately for the chloride, bromide, iodide, and silver ions in water, methyl alcohol, and ethyl alcohol (Koch, J. Chem. Soc. p. 1551 (1930)). Since, however, the basic conditions of Born's formula (and therefore also of Brodsky's formula which may be regarded as merely another form of Born's formula) are hardly ever fulfilled in natural cases, this formula has its chief use in estimating (by comparison with experiment) deviations from the *ideal* (Born) state (Koch, Phil. Mag. (7) x. p. 559 (1930)). The experimental results of section 4 show, in fact, that the dielectric constant is not the only property of the solvent of which the solvation activity coefficient is a function, but that it depends in general upon specific ionic-molecular forces.

(b) *The Walden Formula.*—According to the well-known Walden relation ('Elektrochemie Nichtwässerige Lösungen,' 1924, pp. 355, 443) the solvation activity coefficients of ALL ions in a given solvent should be the SAME, and proportional to the cube of the dielectric constant of the solvent. The experimental results of section 4 show that this supposed relationship receives no experimental confirmation.

§ 6. *Solvation of the Electron.*

Since the radius of the electron is very small its free energy of solvation should, according to Born's formula, be very large. In water, however, the electron is specifically insoluble, *i. e.*, there is no stable hydrated anion of the type $e(H_2O)_x^-$. This conclusion is drawn from the facts, (1) that electronic conduction in aqueous solution is unknown, and (2) that the alkali and alkaline earth metals do not dissolve in water, but react with it, evolving hydrogen, whilst the other electropositive metals (*e. g.*, zinc) are insoluble in this solvent. Since a metal may be regarded as a kind of salt, and since the insolubility of a single

ion of a salt in a given solvent is a condition of the insolubility of the ionized salt in that solvent, it follows that the insolubility of all electropositive metals in water may be attributed to the insolubility of the electron in this liquid. The insolubility of the electron in water is, moreover, an assumption that is implicitly made in the Nernst Theory of Electrode Potentials.

That the electron is soluble in liquid ammonia has, however, been shown by Kraus (J. A. C. S. 1914, xxxvi. p. 864), who concluded the existence of an anion of the type $e(NH_3)_y^-$ from the high transport number of the anion, deduced from measurements of the E.M.F.'s of concentration cells of metallic sodium in liquid ammonia. The solubility of the electron in other solvents of ammoniacal character may be deduced from the solubility of the ionized alkali metals in them (see Kraus, J. A. C. S. xxxix. p. 1557; xxx. pp. 653, 1197, 1323).

In conclusion, the author wishes to express his best thanks to the Trustees of the Beit Fellowship for a Fellowship, and to Dr. H. J. T. Ellingham for criticism.

Physical Chemistry Department,
Imperial College of Science and Technology,
London, S.W. 7.

XLVII. *On some Relationships between Phase Boundary Potential, Adsorption, Surface Tension, and Particle Size.* *By* Frederick Karl Victor Koch, *Ph.D., Beit Scientific Research Fellow* *.

ONLY those interphases are discussed for which the internal constitution of one phase (*e.g.*, an electrode) is constant.

§ 1. *General Considerations.*

The variation of potential difference, e, with surface tension, σ, at an interphase is given by the well-known thermodynamic equation of Lippmann-Helmholtz (Helmholtz, *Gesamm. Abh.* i. p. 925) as

$$\partial e/\partial \sigma = -1/q, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where q is the charge per unit surface.

* Communicated by Prof. James C. Philip, F.R.S.

The variation of adsorption with surface tension at an interphase is given (on the *assumption* of the applicability of the ideal gas laws to solutions or vapours) by the well-known thermodynamic equation of Gibbs ('Collected Papers,' i. p. 233) as

$$a = -C/RT \cdot (\partial\sigma/\partial C), \quad . \quad . \quad . \quad . \quad (2)$$

where a is the number of (gram) molecules adsorbed per unit surface in excess of the number which would be there if the concentration were the same as in the bulk of the solution or in the vapour; C is the concentration of these molecules in the solution (or in the vapour).

Applying (1) to the adsorption of neutral molecules, the charge, q, and potential difference, e, may be regarded as being due to the charges localized in the molecules, which, as is well known, are orientated at the interphase.

Applying (2) to the preferential adsorption of ions, it is evident that a will represent the number of (gram) ions of a given species adsorbed per unit surface in excess of the number which would be there if the concentration were the same as in the bulk of the solution.

Hence, in general,

$$q = (n\mathrm{F}) \cdot a, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $(n\mathrm{F})$ is the charge localized at part of a (gram) molecule or on a (gram) ion, and in the case of ions n is the valency and F the charge per gram equivalent.

By (1), (2), and (3)

$$\partial e/\partial\sigma = (RT/n\mathrm{F}) \cdot (\partial C/\partial\sigma) \cdot 1/C,$$

or, on integration,

$$e = e_1 + (RT/n\mathrm{F}) \cdot \log_e C, \quad . \quad . \quad . \quad . \quad (4)$$

where e_1 is the potential difference when $C = 1$.

Equation (4), which will be recognized as the (Nernst) formula for the variation of electrode potential with the concentration of ions in solutions, is based on the *assumptions* (1) that the ideal gas laws apply to the solution (or vapour), and (2) that the charge at the interphase is directly proportional to the number of ions (or molecules) adsorbed.

§ 2. *Validity of the Helmholtz Condition.* (*Adsorption of Ions.*)

If it be *assumed* (Helmholtz) that the double layer may be regarded as a condenser of *constant* capacity k, then

$$q = k \cdot e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and by (3) and (5)

$$e = (n\mathrm{F}/k) \,.\, a, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

i. e., the interphasial potential difference is directly proportional to the number of ions adsorbed per unit surface.

Inserting (5) in (1) and integrating,

$$\Delta\sigma = \sigma_0 \,.\, -\sigma = (1/2) \,.\, k \,.\, e^2, \quad . \quad . \quad . \quad (7)$$

where σ_0 is the (maximum) surface tension when $e = 0$. (Eq. (7) is the well-known Lippmann parabolic equation of the electro-capillary curve, which is, however, usually written as $\sigma_0 - \sigma = k \,.\, e^2$, because the charge is taken as $2q$.)

By (6) and (7)

$$\Delta\sigma = (n^2\mathrm{F}^2/2k) \,.\, a^2, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

i. e., the change in surface tension is directly proportional to the SQUARE of the number of ions adsorbed per unit surface.

By (4) and (6)

$$e_1 + (\mathrm{RT}/n\mathrm{F}) \,.\, \log_e \mathrm{C} = (n\mathrm{F}/2k) \,.\, a,$$

or

$$a = a_1 + b \,.\, \log_e \mathrm{C}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where a_1 and b are isothermal constants such that $b = \mathrm{C}(\partial a/\partial \mathrm{C})$ and $a = a_1$ when $\mathrm{C} = 1$.

Eq. (9) is therefore the "Adsorption Isotherm." It may be noted that this isotherm has a different mathematical form from that of Freundlich (see later).

By (8) and (9)

$$\Delta\sigma = (n^2\mathrm{F}^2/2k) \,.\, (a_1 + b \,.\, \log_e \mathrm{C})^2$$
$$= \alpha + \beta \,.\, \log \mathrm{C} + \gamma \,.\, (\log \mathrm{C})^2, \quad . \quad . \quad . \quad (10)$$

where α, β, and γ are isothermal constants.

Since eq. (10) gives the variation of surface tension with concentration, it may be termed the "Surface Tension Isotherm."

Eq. (7) has been confirmed experimentally by Krüger and Krumreich (*Z. Elektrochem.* xix. p. 617 (1913)) and others for the adsorption of ions on mercury, showing that the Helmholtz condition is fulfilled at least within certain limits and under certain conditions. Eq. (4) has received ample experimental confirmation. Since no new assumptions have been introduced in deducing equations (6), (8), (9), and (10), it follows that the experimental data that support equations (4) and (7) must also support these other equations.

§ 3. *Non-validity of the Helmholtz Condition.*

The conclusions of the previous section are invalidated in the case of molecules or ions that do not conform to the Helmholtz condition (5). It may be remarked that the Helmholtz assumption of a double layer of constant capacity cannot be expected to hold except in special cases (*e.g.*, for Langmuir monomolecular adsorption layers), on account of its simplicity compared with the complexity of the actual phenomena.

It is, however, known (Freundlich, 'Colloid and Capillary Chemistry,' Eng. transl. p. 111 (1926)) that the *empirical* adsorption formula

$$a = \mathrm{K} \,.\, \mathrm{C}^{1/x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

(where x and K are isothermal constants) is generally applicable to adsorption phenomena, and on the *assumption* of the validity of this formula further conclusions may be drawn.

Eliminating * C between (4) and (11),

$$e = e_1 + x \,.\, (\mathrm{RT}/n\mathrm{F}) \,.\, \log_e a/\mathrm{K},$$

or

$$e = e_1' + x \,.\, (\mathrm{RT}/n\mathrm{F}) \,.\, \log_e a, \quad . \quad . \quad . \quad (12)$$

where e_1' is an isothermal constant, *i.e.*, the interphasial potential difference is a linear function of the logarithm of the number of molecules or ions adsorbed per unit surface (*cf.* (6)).

Eliminating a between (2) and (11), and integrating,

$$\Delta\sigma = \sigma_0 - \sigma = -x \,.\, \mathrm{RT} \,.\, \mathrm{K} \,.\, \mathrm{C}^{1/x} \quad . \quad . \quad . \quad (13)$$

$$= -(x \,.\, \mathrm{RT}) \,.\, a, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

i.e., the change in surface tension is directly proportional to the number of molecules or ions adsorbed per unit surface (*cf.* (10)).

Eq. (13) may be written in the form

$$\Delta\sigma = \mathrm{K}' \,.\, \mathrm{C}^{1/x}, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where $\mathrm{K}' = -x \,.\, \mathrm{RTK}$ is an isothermal constant.

It thus appears that under these conditions the adsorption isotherm and the surface-tension isotherm have the same mathematical form. Eq. (15) has been proposed as an

* It may be noted that Isgarischew (*Z. Elektrochem.* xxxii. p. 281 (1926)) substituted the Freundlich isotherm in the Nernst electrode potential formula; but since he erroneously inserted a in place of C, his further conclusions are invalidated.

empirical formula by Freundlich (*loc. cit.* p. 66), who has shown that the plots of $\log \Delta\sigma$ against $\log$ C are in general not strictly straight, but concave to the log C axis. In this connexion reference may be made to a paper by Kosakavitch (*Z. physikal. Chem.* cxxxiii. p. 1 (1928)) entitled "Über die Oberflächenspannung von nichtwässerige Salzlösungen

§ 4. *Dependence of Phase Boundary Potential on Particle Size.*

It has long been known that the vapour pressure and solubility of a substance increase with decrease in size of the particles of which it is composed. The formula for the increase in vapour pressure with decrease in size of a spherical drop was first calculated by W. Thomson (Phil. Mag. [4] xlii. p. 448 (1881)), who found

$$\mathrm{RT} \,.\, \log_e p_w/p = 2 \,.\, \sigma \mathrm{M}/r\rho,$$

where p_w and p are the vapour pressures of the liquid over a drop of radius r and over the plane surface respectively, σ is the surface tension, ρ is the density, and M the molecular weight of the liquid.

Ostwald (*Z. physikal. Chem.* xxxiv. p. 503 (1900)) applied this formula to the increase in solubility with decrease in particle size by replacing * the vapour pressures p_w and p by the solubilities L_w and L respectively.

An application of Thomson's formula may be made to calculate the change in interphasial potential difference with particle size. In this case *the vapour pressures p_w and p must be replaced by the Nernst solution pressures* P_w *and* P *respectively*. Thus

$$\mathrm{RT} \,.\, \log_e \mathrm{P}_w/\mathrm{P} = 2\sigma\mathrm{M}/r\rho,$$

or

$$\Delta e = (\mathrm{RT}/n\mathrm{F}) \,.\, \log_e \mathrm{P}_w/\mathrm{P} = 2\sigma\mathrm{M}/r\rho n\mathrm{F}, \quad . \quad (16)$$

where Δe is the difference in potential between a spherical particle of radius r and the plane surface and nF is defined by eq. (3).

That this procedure is justified may be seen by the following two proofs, which are the electromotive analogues of the vapour pressure proofs—exactly as the solution pressures are the electromotive analogues of the vapour pressures.

* Actually Ostwald deduced the formula by a somewhat different method, and found the factor 3 instead of 2. Freundlich (*loc. cit.* p. 155) has, however, shown that this is incorrect.

Proof 1.—Consider two spherical particles composed of the same material of density ρ, surface tension σ, and having masses m_1 and m_2, surface areas w_1 and w_2, radii r_1 and r_2, and potential differences at their surfaces e_1 and e_2 respectively.

Let a gram ion, mass M, be transferred from the first to the second particle; then the surface work done is

$$\sigma(dw_1-dw_2).$$

Now

$$dw=\mathrm{M}\,.\,(\partial w_1/\partial m_1), \quad dw_2=\mathrm{M}\,.\,(\partial w_2/\partial m_2),$$

and for an isotropic sphere, in general,

$$\partial w/\partial m=2/r\rho,$$

and therefore the surface work is

$$2\sigma\mathrm{M}/\rho(1/r_1-1/r_2).$$

Since the electrical work is

$$n\mathrm{F}(e_1-e_2)=n\mathrm{F}\Delta e,$$

$$\Delta e=2\sigma\mathrm{M}/\rho n\mathrm{F}\,.\,(1/r_1-1/r_2),$$

and putting $r_2=\infty$ (*i.e.*, for a plane surface) and $r_1=r$, we have

$$\Delta e=2\sigma\mathrm{M}/r\rho n\mathrm{F}.$$

Proof 2.—Consider the gravity cell depicted in the diagram. The darkly shaded portions represent mercury

and the lightly shaded portions a solution of a mercury salt. The meniscus C is one electrode and the plane surface P is the other electrode.

Owing to the action of gravity the osmotic pressure of the ions in solution at C (p_C) will be greater than at P (p_P),

and if the potential differences at the interphases at C and P are e_C and e_P respectively, then, by application of the original Nernst formula,

$$e_C = (RT/nF) \cdot \log_e P_C/p_C,$$
$$e_P = (RT/nF) \cdot \log_e P_P/p_P,$$

where P_C and P_P are the solution pressures of mercury at C and P respectively.

But the condition of equilibrium is

$$e_C - e_P = 0,$$

whence

$$P_C/P_P = p_C/p_P,$$

and since $p_C > p_P$, it follows that $P_C > P_P$.

By application of the hypsometric formula to the mercury ions in solution

$$\text{h.g.M.}(1-\rho_S/\rho_M) = RT \cdot \log_e p_C/p_P$$

(where M is the weight of a gram mercury ion, ρ_M is the density of mercury, and ρ_S the mean density of the solution), and by a consideration from the point of view of surface tension

$$2 \cdot \pi \cdot r \cdot \sigma = \pi \cdot r^2 \rho_M \cdot h \cdot (1-\rho_S/\rho_M) \cdot g$$

(where r is the radius of the capillary tube and also of the curvature of the mercury meniscus at C). Whence

$$(RT/nF) \cdot \log_e p_C/p_P = 2 \cdot \sigma \cdot M/r\rho_M nF,$$

and therefore

$$(RT/nF) \cdot \log_e P_C/P_P = 2\sigma M/r\rho_M nF.$$

But $(RT/nF) \cdot \log_e P_C/P_P$ is the E.M.F. which the cell would have if $p_C = p_P$ (*i. e.*, when $h=0$).

Hence, putting $\rho_M = \rho$,

$$\Delta e = 2\sigma M/r\rho nF.$$

The above considerations are general, and apply to all interphasial potential differences. In the case of ions, if the ion is positive, the particle will be negative relative to the plane surface, and *vice versa*.

For mercury (univalent) drops in water at 20° $n = 1$, $M = 200{\cdot}6$ grams, $\rho = 13{\cdot}546$ grams/cm., $\sigma = 375$ dynes/cm., and $F = 96{,}500$ coulombs; whence

$$\Delta e = 1{\cdot}205 \times 10^{-11}/r \text{ volt},$$

when

$r = 1000\ \mu\mu$, $\Delta e =$ 1·2 microvolts.
„ = 100 „ „ = 12 „
„ = 10 „ „ = 120 „

Formula (16) (which, so far as the author is aware, has not been deduced previously) is of possible interest in connexion with the electrodeposition of metals and the reproducibility of electrode potentials etc., and, in fact, surface chemistry generally.

Physical Chemistry Department,
Imperial College of Science and Technology,
London, S.W.7.

XLVIII. *The Density of White Dwarf Stars.*

By S. Chandrasekhar *.

1. THE first application of the Fermi-Dirac statistics to stellar problems was by Fowler † in connexion with the well-known problem of the companion of Sirius. This idea has lately been taken up by Stoner ‡ and others to calculate the limiting density of white dwarf stars. In this paper another way of arriving at the order of magnitude of the density of white dwarfs from different considerations is given.

2. Let p_r denote the radiation pressure and p_G the gas pressure, and the total pressure P is then given by

$$P = p_r + p_G. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We introduce the constant β, such that

$$\left.\begin{aligned} p_r &= (1-\beta)P, \\ p_G &= \beta P. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We will make the assumption that $\beta = 1$ approximately, *i.e.*, we leave the radiation pressure out of account. We are dealing therefore with *ideal* conditions which can perhaps exist only in stars which are much higher in the white dwarf stage than even O_2, Eridani B.

* Communicated by R. H. Fowler, F.R.S.
† R. H. Fowler, Month. Not. Roy. A. S. lxxxvii. p. 114 (1926).
‡ E. C. Stoner, Phil. Mag. vii. p. 63 (1929); ix. p. 944 (1930).

Now for a fully degenerate electron gas (in the Sommerfeld sense) the pressure is given by

$$p_e = \frac{\pi}{60}\frac{h^2}{m}\left(\frac{3n}{\pi}\right)^{5/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We assume that it is this electron pressure which is by far the greatest contribution to the gas pressure, and therefore to the total pressure. Further, if ρ is the density of the stellar material, the number of electrons is given by

$$n = \frac{\rho}{\mu \mathrm{H}(1+f)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where f is the ratio of the number of ions to the number of electrons (we can in practice neglect f), H the mass of the hydrogen atom, and μ the molecular weight. For a fully ionized material of the type we are dealing with $\mu = 2{\cdot}5$ nearly. We will use this value later. We have therefore

$$p_G = \frac{\pi}{60}\frac{h^2}{m}\left(\frac{3}{\pi \mathrm{H}}\right)^{5/3}\frac{\rho^{5/3}}{\mu^{5/3}(1+f)^{5/3}}$$

$$= 9{\cdot}845 \times 10^{12}\left[\frac{\rho}{\mu(1+f)}\right]^{5/3}, \quad . \quad . \quad . \quad (5)$$

(The values used for h, m, etc. are those given in A. S. Eddington's 'Internal Constitution of Stars,' Appendix (1).)

Putting

$$\mathrm{K} = \frac{9{\cdot}845 \times 10^{12}}{\mu^{5/3}(1+f)^{5/3}}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we have for the total pressure

$$\mathrm{P} = \mathrm{K}\rho^{5/3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We can now straightway apply the theory of the polytropic gas spheres, where for the exponent γ we have

$$\gamma = 5/3 \text{ or } 1 + \frac{1}{n} = 5/3,$$

giving

$$n = 3/2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

We have therefore *

$$\left(\frac{\mathrm{GM}}{\mathrm{M}'}\right)^{+1/2}\left(\frac{\mathrm{R}'}{\mathrm{R}}\right)^{-3/2} = \frac{[5/2\mathrm{K}]^{3/2}}{4\pi\mathrm{G}}, \quad . \quad . \quad . \quad (9)$$

* A. S. Eddington, 'Internal Constitution of Stars,' p. 83 *et seq.* The notation is the same as that used in his book and now generally adopted. The particular equation (9) follows from the second of the equations (57.3).

$$\text{or} \qquad \frac{GM}{M'} = \frac{125 \times 9{\cdot}845^3 \times 10^{36}}{128\pi^2 G^2} \cdot \frac{1}{\mu^5(1+f)^5}\left(\frac{R'}{R}\right)^3. \quad . \quad (10)$$

The values of R′ and M′ can be obtained from the extensive tables given by Emden in his 'Gas-Kugeln,' and are (page 79, tabbelle 4)

$$\left.\begin{aligned} R' &= 3{\cdot}6571, \\ M' &= 2{\cdot}7176. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Using these values in (10), and expressing the mass in terms of that of the Sun ($= 1{\cdot}985 \times 10^{33}$ grams), we get the result

$$(M/\odot)R^3 = \frac{2{\cdot}14 \times 10^{28}}{\mu^5} (= 2{\cdot}192 \times 10^{26}). \quad . \quad (12)$$

The second value for $(M/\odot)R^3$, given in brackets, we get by using the value 2·5 for μ. We can express (12) differently, as follows:

$$R^6\rho = \frac{1{\cdot}014 \times 10^{61}}{\mu^5} (= 1{\cdot}039 \times 10^{59}), \quad . \quad . \quad (13)$$

$$\rho = 2{\cdot}162 \times 10^6 (M/\odot)^2. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

We will apply the above equations to the case of the companion of Sirius. The mass of it, as determined from the double star orbit, is trustworthy, and equals ·85⊙. The computed radius $= 1{\cdot}8 \times 10^7$. (But we cannot use this value in (13) to calculate the density, as it is based on formulæ which may not be applicable to this case.) From the mass we can derive the radius and equals $6{\cdot}361 \times 10^8$ (about thirty times the accepted value). For the density of the companion of Sirius we get from (14), *provided* it were completely degenerate (which, however, is extremely unlikely),

$$\rho_{\text{C. Sirius}} = 1{\cdot}562 \times 10^6 \text{ grams per cm.}^3 \quad . \quad (15)$$

The mean density assumed is $\cdot 5 \times 10^5$, being thus thirty times smaller than that given by (15). We can, however, take the value given by (15) as indicating the *maximum* density which a stellar material having a mass equal to that of the companion of Sirius can have. A similar calculation can be made for O_2 Eridani B and Procyon B, and the calculated values are collected in a table below. The calculations for the *limiting density* on Stoner's theory give different values, and they are also given for comparison. We discuss the cause of the difference below.

We further note (i.) that the radius of a white dwarf is inversely proportional to the cube root of the radius, (ii.) the density is proportional to the square of the mass, (iii.) the central density would be six times the mean density ρ.

3. Stoner (*loc. cit.*) arrives at a formula for the *limiting* density for a material composed of completely ionized atoms on the following argument:—

The density increases as the sphere shrinks, and the limit is reached when the gravitational energy released just supplies the "energy required to squeeze the electrons closer together." The limiting condition would then be given by

$$\frac{d}{dn}(E_G + E_K) = 0, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

E_G being the gravitational energy and E_K the kinetic energy, for which, of course, the Fermi formula is used. The formula he gets is (without his latter relativity-mass correction)

$$\rho_{max.} = 3{\cdot}977 \times 10^6 (M/\odot)^2, \quad . \quad . \quad . \quad . \quad (17)$$

which is exactly the same as our (14) with a difference in the

Star.	Mass.	Radius.	Density.		
			As calc. by (14).	Accepted value.	By Stoner's formula (17).
O_2 Eridani B.	·44 ⊙	$7{\cdot}927 \times 10^8$	$4{\cdot}186 \times 10^5$	$·98 \times 10^5$	$7{\cdot}8 \times 10^5$
Procyon B.	·37 ⊙	$8{\cdot}399 \times 10^8$	$2{\cdot}960 \times 10^5$	—	$5{\cdot}445 \times 10^5$
Companion of Sirius.	·85 ⊙	$6{\cdot}361 \times 10^8$	$1{\cdot}562 \times 10^6$	$·5 \times 10^5$	$2{\cdot}872 \times 10^6$

numerical factor only, the discrepancy being about 1 : 2. The difference in the two is obviously due to the fact that our value for ρ is not the "limiting density" in the sense in which Stoner uses the term ; but our calculation gives us a much nearer approximation to the conditions actually existent in white dwarfs than Stoner's calculation does. At any rate, it brings out clearly that the *order of magnitude* of the density which one can on purely theoretical considerations attribute to a white dwarf is the same.

Our results (see table) agree with Stoner's in showing that O_2 Eridani B is much nearer the ideal dwarf-star stage than the companion of Sirius, but indicate also that neither of them is so far from the ideal stage as Stoner's calculation would seem to indicate.

Summary.

The density of the white dwarf stars is reconsidered from the point of view of the theory of the polytropic gas spheres, and gives for the *mean density* of a white dwarf (under ideal conditions) the formula

$$\rho = 2{\cdot}162 \times 10^6 \times (\mathrm{M}/\odot)^2.$$

The above formula is derived on considerations which are a much nearer approximation to the conditions *actually existent* in a white dwarf than the previous calculations of Stoner based on uniform density distribution in the star and which gave for the limiting density the formula

$$\rho = 3{\cdot}977 \times 10^6 \times (\mathrm{M}/\odot)^2.$$

XLIX. *The Magnetization-Temperature Curves of Iron, Cobalt, and Nickel.* *By* F. Tyler, *B.Sc., A.Inst.P.* *

Introduction.

In a previous paper [1] the theoretical significance of the spontaneous magnetization-temperature curves for nickel has been discussed. On the Weiss theory, the spontaneous magnetization of a ferromagnetic is determined by the equations

$$\frac{\sigma}{\sigma_0} = \coth a - \frac{1}{a}, \qquad (1)$$

$$a = \frac{\sigma \mathrm{H}}{\mathrm{RT}},$$

$$\mathrm{H} = \mathrm{H}_e + \mathrm{NI}, \qquad (2)$$

where σ is the gram. molecular moment, σ_0 the gram. molecular saturation moment, I the intensity of magnetization, and N the coefficient of the molecular field. The spontaneous magnetization which appears below a certain temperature (the Curie temperature) for $\mathrm{H}_e = 0$ can be obtained as a function of the temperature by a graphical process.

In the quantum modification of this theory there is a change in the distribution function, and equation (1) is replaced by the general expression

$$\frac{\sigma}{\sigma_0} = \frac{1}{j} \sum_{-j}^{j} e^{mgh} \cdot m \Big/ \sum_{-j}^{j} e^{mgh}, \qquad (3)$$

* Communicated by Prof. R. Whiddington, F.R.S.

where m, j, g have their usual significance as in spectroscopic theory and $h=\frac{\mu_B H}{kT}$ ($\mu_B=1$ Bohr magneton).

Spontaneous magnetization curves have been constructed from (3) with $j=\frac{1}{2}$, 1, $1\frac{1}{2}$, and for nickel it was found that there was very close agreement between the experimental curve and the theoretical curve for $j=\frac{1}{2}$. For $j=\frac{1}{2}$ equation (3) takes the form

$$\frac{\sigma}{\sigma_0}=\tanh a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This result is in conformity with that obtained from the Heisenberg [2] interchange interaction theory of ferromagnetism when certain simplifying assumptions are made [13, 14].

It was thought to be of further interest to construct "corresponding state" curves for iron and cobalt from the data available for these metals, and to compare the experimental with the theoretical curves, as was done for nickel.

Experimental Data.

In constructing the corresponding state curve for nickel use was made of the recent precise measurements of Weiss and Forrer [3]. As far as the writer is aware no comprehensive results of comparable accuracy have been published for iron and cobalt. The general character of the curves is, however, given satisfactorily by the experimental results of Curie [4] for iron, and of Bloch [5] for cobalt. Accurate low-temperature results for iron are provided by further recent work of Weiss and Forrer [6]. Supplementary data for iron, cobalt, and nickel are obtainable from the researches of Honda and Shimizu [7] on magnetization of ferromagnetics.

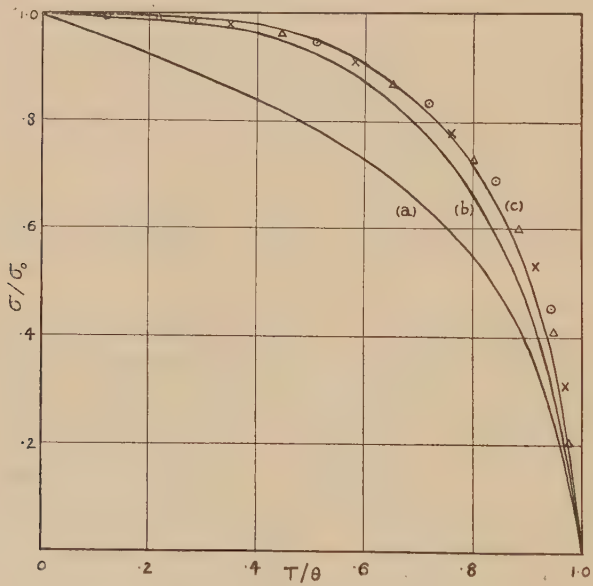
In the experiments measurements are made of the magnetic intensity for various values of an applied field, the temperature being maintained constant. To obtain the quasi-saturation moment at any particular temperature it is therefore necessary to extrapolate the magnetic isothermals to zero external field. For temperatures well below the Curie temperature extrapolation is reasonably certain and presents little difficulty. In this region and towards the absolute zero a law of approach given by Weiss may be used :

$$\sigma_{T,H}=\sigma_{T,\infty}\left(1-\frac{a}{H}\right), \quad . \quad . \quad . \quad . \quad (5)$$

where $\sigma_{T,H}$ is the magnetization in a field H at T° and

$\sigma_{T,\infty}$ its limiting value. a is a constant giving a measure of the magnetic hardness in strong fields. For temperatures nearer the Curie temperature the extrapolation process becomes more uncertain. The shape of the curve in this region, however, may be obtained by considering curves relating σ to T for external fields of successively decreasing magnitudes.

The final curves obtained for iron and cobalt and that of nickel are given, along with classical and quantum theoretical



Spontaneous magnetization curves.

(*a*) Classical curve (Langevin).
(*b*) Quantum curve, $j=1$.
(*c*) Quantum curve, $j=\frac{1}{2}$.

⊙⊙⊙, Iron. ×××, Cobalt. ΔΔΔ, Nickel.

curves, in the figure. Table I. gives the numerical values of $\frac{\sigma}{\sigma_0}$ for the three metals tabulated against $\frac{T}{\theta}$. Theoretical values of $\frac{\sigma}{\sigma_0}$ are also given, so as to show clearly the degree of correspondence between experiment and theory. Table II. collects the most reliable recent values of the magnetic constants for the three ferromagnetic metals.

TABLE I.

Spontaneous magnetization values.

T/θ	σ/σ_0. Classical theory.	Quantum theory. $j=1$.	Quantum theory. $j=\frac{1}{2}$.	Experimental. Fe.	Co.	Ni.
0	1·00	1·00	1·00	1·00	1·00	1·00
·1	·97	$\cdot 99_7$	$\cdot 99_8$	$\cdot 99_5$	$\cdot 99_5$	$\cdot 99_5$
·2	·93	$\cdot 99_3$	$\cdot 99_5$	$\cdot 98_5$	$\cdot 98_5$	·99
·3	·89	·99	$\cdot 99_2$	$\cdot 97_5$	·97	·98
·4	·84	·97	·98	·96	·95	·96
·5	·79	·93	·96	$\cdot 93_5$	·92	·93
·6	·73	·88	·91	·91	·89	·89
·7	·66	·80	$\cdot 83_5$	·85	$\cdot 83_5$	$\cdot 83_5$
·8	·55	·67	·72	·76	$\cdot 73_5$	·73
·9	·40	·49	·52	·60	$\cdot 57_5$	·57
1·0	0	0	0	0	0	0

TABLE II.

Magnetic constants for the ferromagnetic metals.

	Fe (26).	Co (27).	Ni (28).
A	55·84	58·97	58·68
ρ	7·86	8·6	8·9
σ_0	211·79	161·50	57·60
p	11	8·46	3
θf	758	1388	357·6
θp	774	1411	376

A. Atomic weight.
ρ. Density.
σ_0. Saturation moment per gramme.
p. Weiss magneton number corresponding to σ_0.
θf. Ferromagnetic Curie point in °C.
θp. Paramagnetic Curie point in °C.

The number in brackets following each element is the atomic number.

Discussion.

A study of the figure and Table I. shows quite definitely that the corresponding state curves for iron and cobalt are almost identical with one another and also with the curve for nickel. Further, they are definitely incompatible with the classical curve or with the quantum curves for j equal to or greater

than 1. For the greater part of the range, however, there is close agreement with the $j=\frac{1}{2}$ curve. The divergence near the Curie point is in part due to the experimental difficulties and also to the uncertainty in extrapolation in this region. The low-temperature results cannot be quite reconciled with theory—the approach to saturation being slightly slower than that indicated by a tanh curve. A formula given by Bloch [8] in this connexion, viz.,

$$\frac{\sigma}{\sigma_0}=1-\left(\frac{\mathrm{T}}{\theta}\right) \quad , \quad . \quad . \quad . \quad . \quad . \quad (6)$$

gives an approach to saturation which is little better than that derived from the classical function of Langevin, and hence cannot claim to be in agreement with the experimental results. The important point to be observed, however, is that over the major part of the range the $j=\frac{1}{2}$ curve is in close conformity with the experimental results—thus indicating a tanh distribution.

Hence to a close approximation the magnetic elements for all the three ferromagnetic metals correspond to magnetic carriers for which $j=\frac{1}{2}$. This conclusion has an important bearing on the question as to the nature of the elementary magnetic carriers. A j value of $\frac{1}{2}$ is consistent with the view that the carriers are free electrons or atoms or ions in which the s moment only is operative. Now experiments on the gyromagnetic effect show conclusively that in ferromagnetics at ordinary temperatures the ratio of the magnetic to the mechanical moment of the carriers is 2 (both being measured in Bohr units). Hence a g value of 2 taken in conjunction with a j value of $\frac{1}{2}$ from above indicates that essentially it is the electron spin which is responsible for the elementary magnetic moment.

The problem as to the state of these magnetic electrons has, however, provoked much discussion. The first question which arises is whether the electrons are free (*i. e.*, capable of forming an electron gas in the Sommerfeld sense) or bound. Experiments on the thermoelectric properties of ferromagnetics performed by Dorfman [9] and his co-workers on nickel were at first interpreted as indicating that the magnetic carriers for this metal were also the conduction electrons (and so, if Sommerfeld's theory is adopted, free electrons), but a reconsideration of the results has shown that this conclusion does not follow. A possible alternative explanation has recently been suggested by Stoner [10].

Arguments against the view that the magnetic carriers are free electrons are not lacking [11]. Stoner [12] shows that

if this assumption is made then the energy values required to produce spontaneous magnetization would be such as to lead to Curie temperatures of the order of many thousands of degrees (*e.g.*, for nickel θ_c would be $>43{,}160^\circ$ K.). The conclusion is accordingly made that the electrons responsible for ferromagnetism are the interchange electrons of Heisenberg's theory.

Conclusion.

In all probability, therefore, it is electron spin which is to be regarded as the fundamental or elementary magnetic element operative in the three ferromagnetic metals iron, cobalt, and nickel. The spin, however, is not associated with a free electron, but with an electron bound in an atom or ion—the ferromagnetic effects arising from some interaction process between the spins in neighbouring ions. Heisenberg's theory of ferromagnetism incorporates these features, and in so far as the broad outline of the theory is concerned the experimental results are in excellent agreement with it provided that a suitable number of interacting electrons is chosen in each case—in nickel less than one per atom, and in cobalt and iron about two and three respectively [13].

Perhaps the most important conclusion emerging from the above consideration of the experimental results is the fact that the spontaneous magnetization curve for $j=\frac{1}{2}$ is as applicable to iron and cobalt, which certainly have more than one interacting electron per atom, as to nickel, which has less than one. A value of $\frac{1}{2}$ for j must indicate that the electron spins are independent; if, for example, the two or more spin moments per atom in iron were coupled to give a resultant spin in the usual way, values $>\frac{1}{2}$ would be observed. The change of specific heat at the Curie point also seems to necessitate an effective independence of the electron spin moments [14]. It may be noted here that electron spin associated with ions may arise owing to defect as well as excess of electrons necessary to produce a completed configuration. In the ferromagnetic elements it is probably defect rather than excess that is operative.

Summary.

Reference is made to the modification produced in Langevin's distribution function by quantum considerations.

The experimental corresponding state curves for iron and cobalt are compared with the theoretical curves obtained from the classical and quantum functions, as had previously been done for nickel. It is observed that the curves for all

three ferromagnetic metals are closely coincident, and for a greater part of the range are practically coincident with the quantum curve for $j=\frac{1}{2}$. (This indicates a tanh distribution.) The bearing of the evidence on the question of the nature and state of the elementary carriers is discussed. It is concluded that these are electrons with independent spin moments. The electrons are not free, however, but are associated with ions as in Heisenberg's theory.

The apparent independence of the spin moments of the carriers in iron and cobalt (which certainly contain more than one effective magnetic electron per atom) is stressed.

Finally, I wish to record my thanks to Dr. E. C. Stoner for valuable discussion, and my indebtedness to the Department of Scientific and Industrial Research for a maintenance grant.

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Physics Laboratories,
University of Leeds.
Oct. 1930.

L. *The Stark Effect in Band Spectra.*

By W. G. Penney, *A.R.C.S., B.Sc.* *

Introduction.

MEASUREMENTS of the Stark effect have been almost entirely confined to atomic spectra. This has been due to the fact that, on the one hand, the production of such

* Communicated by Dr. R. de L. Kronig.

spectra in the presence of a strong electric field is much easier than that of molecular spectra, and that, on the other hand, in the former case the interpretation was more complete and in consequence the theoretical interest greater. With the increasing importance which the subject of band spectra has taken on in recent years it seems worth while to investigate what effect an electric field may be expected to have on molecular spectra, in the hope that the experimental difficulties involved in the testing of the results may soon be overcome.

The effect of an electric field on a molecule is most conveniently determined by means of the usual perturbation method[1], according to which the energy W_j of the state j in the presence of the field is developed in powers of the field strength E:

$$W_j = W_j^0 + W_j^1 E + W_j^{(2)} E^2 + \ldots,$$

where W_j^0 is the unperturbed energy value

$$W_j^1 = H'(j, j), \quad \ldots \quad (1)$$

$$W_j^{(2)} = \sum_{j'}{}' \frac{H'(j, j') H'(j', j)}{h\nu(j, j')}, \quad \ldots \quad (2)$$

while $W^{(3)}$, $W^{(4)}$ are given by more complicated expressions, which we do not put down here, as we shall confine our attention to terms linear and quadratic in E. H' is the perturbation function which, with the electric field in the z direction, is given by the negative of the electric moment in that direction:

$$H' = -\sum_r e_r z_r,$$

where the sum extends over all the particles of the molecule with charges e_r. The summation in (2) extends over all states

$$j' \neq j.$$

From (1) it appears that the linear Stark effect of the state j may be calculated from properties of this state alone, while the evaluation of the quadratic Stark effect involves a knowledge of the other states j'. Among these the most important will be the states for which $\nu(j, j')$ is small, or, in other words, those whose energies lie sufficiently close to that of the given state. Now, according to Born and Oppenheimer[2], in the case of the general polyatomic molecule the motion, just as in the well known case of the diatomic molecule, may be regarded as a superposition of electronic motion, nuclear vibration, and nuclear rotation. Taking H' as being at most of the order of magnitude 10^{-18} (a fraction

of the electronic charge × atomic dimensions), it appears that for a given level j in general only the other rotational levels of the same electronic and vibrational state of the molecule lie sufficiently close to give a measurable contribution to the energy change (2). As regards the positions of such a set of rotational levels in the absence of the field and the dependence of their wave functions on the variables specifying the rotation, the molecule, according to Born and Oppenheimer (*loc. cit.*), may be regarded as a rigid body with principal moments of inertia equal to the mean moments of inertia of the molecule in the given electronic and vibrational state. We hence proceed to calculate the Stark effect up to terms in E^2 for the first few rotational levels of a rigid body with three different moments of inertia and an electric moment. Furthermore, we give an expression valid for all rotational states in the case where two moments of inertia are equal or nearly equal. Finally, the Stark effect in diatomic molecules is discussed.

Stark Effect in Lower Rotational Levels of the Asymmetrical Top.

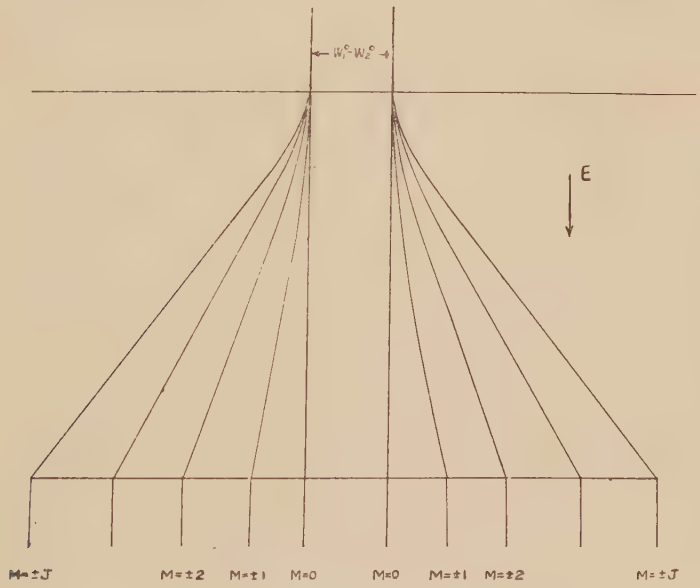
The wave mechanics of the asymmetrical top has been investigated by Kramers and Ittmann[3] and by Wang[4]. They find that its stationary states can be characterized by three quantum numbers, J, n, and M. J measures the total moment of momentum. For every value of J there exist in general $(2J+1)$ distinct states characterized by the quantum number n. In accordance with the notation of Kramers and Ittmann we let n run from $-J$ to $+J$, $n=-J$ being the lowest and $n=+J$ the highest of the levels J. Each of them has a $(2J+1)$-fold degeneracy, and can be split in an external field into $(2J+1)$ component states characterized by the value M of the angular momentum in the direction of the field. For the general asymmetrical case treated in this section, in which the moments of inertia actually are different, the labour involved in the calculations increases rapidly with J. At the same time the experimental difficulties involved in measuring the Stark effect increase as J gets larger, the total splitting becoming less and the number of component levels greater. We shall therefore confine ourselves to calculating the Stark effect for the levels $J=0$ and $J=1$.

It appears that the matrix elements of the perturbation matrix $H'(j, j')$ are different from zero only if the two states j and j' have values of J differing by not more than unity and equal values of M. In calculating the Stark effect of

the levels $J = 0$ and $J = 1$ up to terms in E^2 it is hence, according to equations (1), (2), sufficient to know the unperturbed energy values and wave functions of the levels up up to and including $J = 2$. They have been given by the above authors, and are repeated in the table on p. 606.

Here $a > b > c$ are the reciprocals of the three principal moments of inertia, while $x_1 > x_2$ are the two roots of the quadratic equation

$$\frac{1}{a-x} + \frac{1}{b-x} + \frac{1}{c-x} = 0.$$



θ, ϕ, ψ are Euler's angles specifying the orientation of the principal axes ξ, η, ζ of the top relative to the axes x, y, z fixed in space (θ is the angle between the z- and ζ-axes, where $0 \leq \theta \leq \pi$; the positive direction of the line of nodes is so chosen that in going from the z- to the ζ-axis through the angle θ we turn in the positive sense around it; ϕ is the angle from the nodal line to the ξ-axis measured in the positive sense around the ζ-axis; ψ is the angle from the x-axis to the line of nodes measured in the positive sense around the z-axis). The functions $\Theta_{J\Omega M}(\theta, \psi)$ are the wave functions occurring in the problem of the symmetrical top as found by

Reiche and Rademacher[5] and by Kronig and Rabi[6]. They are given by[7]

$$\Theta_{J\Omega M}(\theta,\psi)=\left[\frac{(-1)^{\alpha+\beta-\gamma}(\alpha-\beta)\,(\alpha-1)!\,(\gamma-\beta-1)!}{2\pi\,2^{\alpha+\beta}(-\beta)!\,[(\gamma-1)!]^2(\alpha-\gamma)!}\right]^{\frac{1}{2}}$$
$$\times(1+\cos\theta)^{\frac{\gamma-1}{2}}(1-\cos\theta)^{\frac{\alpha+\beta-\gamma}{2}}\mathrm{F}\left[\alpha,\beta,\gamma,\tfrac{1}{2}(1+\cos\theta)\right]e^{iM\psi},$$

J.	n.	$\frac{8\pi^2}{h^2}W_{Jn^0}$.	U_{JnM}.
0	0	0	$\frac{1}{2\pi\sqrt{2}}$
1	−1	$b+c$	$\frac{1}{2\sqrt{\pi}}[\Theta_{11M}e^{i\phi}-\Theta_{1-1M}e^{-i\phi}]$
	0	$a+c$	$\frac{1}{2\sqrt{\pi}}[\Theta_{11M}e^{i\phi}+\Theta_{1-1M}e^{-i\phi}]$
	1	$a+b$	$\frac{1}{\sqrt{2\pi}}\Theta_{10M}$
2	−2	$4(a+b+c)-6x_1$	$\frac{1}{\sqrt{2\pi}}[S_0\Theta_{20M}+S_2(\Theta_{22M}e^{2i\phi}+\Theta_{2-2M}e^{-2i\phi}[$
	−1	$a+b+4c$	$\frac{1}{2\sqrt{\pi}}[\Theta_{22M}e^{2i\phi}-\Theta_{2-2M}e^{-2i\phi}]$
	0	$a+4b+c$	$\frac{1}{2\sqrt{\pi}}[\Theta_{21M}e^{i\phi}-\Theta_{2-1M}e^{-i\phi}]$
	1	$4a+b+c$	$\frac{1}{2\sqrt{\pi}}[\Theta_{21M}e^{i\phi}+\Theta_{2-1M}e^{-i\phi}]$
	2	$4(a+b+c)-6x_2$	$\frac{1}{\sqrt{2\pi}}[S_0'\Theta_{20M}+S_2'(\Theta_{22M}e^{2i\phi}+\Theta_{2-2M}e^{-i\phi})]$

where

$$\alpha=\tfrac{1}{2}|M+\Omega|+\tfrac{1}{2}|M-\Omega|+J+1,$$
$$\beta=\tfrac{1}{2}|M+\Omega|+\tfrac{1}{2}|M-\Omega|-J,$$
$$\gamma=\tfrac{1}{2}|M+\Omega|+1,$$

and where F is the hypergeometric function. We note the relationship

$$\Theta_{J\Omega M}(\pi-\theta,\psi+\pi)=(-1)^{J-\Omega}\Theta_{J-\Omega M}(\theta,\psi),$$

useful in the integrations to be carried out when calculating the matrix elements of H′. The coefficients S in the table are given by

$$S_0 = \frac{k\sqrt{6}}{2\sqrt{1+3k^2}-\sqrt{1+3k^2}}, \qquad S_2 = \frac{1-\sqrt{1+3k^2}}{2\sqrt{1+3k^2}-\sqrt{1+3k^2}},$$

$$S_0' = \frac{k\sqrt{6}}{2\sqrt{1+3k^2}+\sqrt{1+3k^2}}, \qquad S_2' = \frac{1+\sqrt{1+3k^2}}{2\sqrt{1+3k^2}+\sqrt{1+3k^2}},$$

$$k = \frac{a-b}{a+b-2c}.$$

If the top has electric moments (p, q, r) in the directions of the principal axes of inertia, then the perturbation function has the form

$$H' = p\sin\theta\sin\phi + q\sin\theta\cos\phi + r\cos\theta.$$

Calculating the various matrix elements of H′ with the help of the wave functions given in the table, and substituting these as well as the frequencies calculated from the energy values given there in equations (1) (2), we obtain the following expressions for the perturbed energy levels :—

$$W_{000} = -\frac{8\pi^2}{h^2}\frac{E^2}{3}\left[\frac{p^2}{b+c}+\frac{q^2}{c+a}+\frac{r^2}{a+b}\right],$$

$$W_{1-1M} = \frac{h^2}{8\pi^2}(b+c)+\frac{8\pi^2}{h^2}E^2\left[\frac{p^2}{3(b+c)}-\frac{q^2M^2}{4(a-c)}-\frac{r^2M^2}{4(a-b)}\right.$$

$$\left.-\frac{q^2(4-M^2)}{20(a+3c)}-\frac{r^2(4-M^2)}{20(a+3b)}-\frac{4qr^2(4-M^2)}{15(8ab+6bc+8ca-3b^2-3c^2)}\right],$$

with

$$M = -1,\ 0,\ 1.$$

W_{10M} is obtained from W_{1-1M} by replacing b by a and q by p, and W_{11M} is obtained by replacing c by a and r by p in W_{1-1M}. As one sees, the effect is quadratic in E and lies near the limit of measurement.

Case of Slight Asymmetry.

If two of the moments of inertia of the top are nearly equal, then the energy levels with a given J will occur in pairs excepting one level, such that the energy difference between each pair is small compared with the remaining energy differences. In investigating the effect of an electric field in this case we may in approximation retain only those elements of the perturbation matrix belonging to such a pair

of states and neglect the other elements. Then the perturbation can be calculated rigorously and not only to terms in E^2.

If a is very nearly equal to b, then the levels $n = 2l - J$ and $n = 2l + 1 - J$ almost coincide. If, on the other hand, b nearly equals c, then the levels $n = 2l - 1 - J$ and $n = 2l - J$ almost coincide ($l = 0, 1, \ldots, J$). In the first case we denote $(J - l)$ by Ω, and in the second case we denote l by Ω. According to Wang (*loc. cit.*) the energy difference $W_1^0 - W_2^0 > 0$, of two such states forming a doublet in the absence of the electric field is in the first case

$$W_1^0 - W_2^0 = k^\Omega h^2 (a + b - 2c)(J + \Omega)!/\pi^2 2^{3\Omega+1}(J - \Omega)![(\Omega - 1)!]^2, \quad (3)$$

while the wave functions with sufficient approximation are given by

$$\frac{1}{2\sqrt{\pi}}[\Theta_{J\Omega M} e^{i\Omega\phi} \pm \Theta_{J-\Omega M} e^{-i\Omega\phi}]. \quad . \quad . \quad . \quad (4)$$

In the case where b is nearly equal to c, we must interchange in equation (3) a and c, while the meaning of the angles in equation (4) is different in so far as θ is now measured between the z- and ξ-axes.

According to the perturbation theory the energy values W in the presence of the electric field are the roots of the quadratic equation

$$\begin{vmatrix} W - W_1^0 & H_{21}' \\ H_{12}' & W - W_2^0 \end{vmatrix} = 0,$$

where H_{12}' is the matrix element of H′ belonging to the states 1 and 2 and H_{21}' its conjugate. The solution of the equation gives

$$\left.\begin{aligned} W_{1M} &= \tfrac{1}{2}\left[W_1^0 + W_2^0 + \sqrt{(W_1^0 - W_2^0)^2 + 4\left\{\frac{r E \Omega M}{J(J+1)}\right\}^2}\right], \\ W_{2M} &= \tfrac{1}{2}\left[W_1^0 + W_2^0 - \sqrt{(W_1^0 - W_2^0)^2 + 4\left\{\frac{r E \Omega M}{J(J+1)}\right\}^2}\right], \end{aligned}\right\} \quad (5)$$

in the case $a \doteq b$, while p takes the place of r if $b \doteq c$.

The general behaviour of the two levels for increasing field is indicated in the diagram. Each level splits into $(J + 1)$ components, the level $M = 0$ not being displaced. The levels with M different from zero have each the weight 2, M and $-M$ coinciding. At first the Stark effect is quadratic; for stronger fields it becomes linear in E. If two moments of inertia are exactly equal, the linear effect sets in right at the beginning.

An example of a molecule behaving like a (nearly) symmetrical top is NH_3. Also a diatomic molecule may be regarded as a slightly asymmetrical top ; for, neglecting at first the small masses of the electrons, the moments of inertia about axes at right angles with the internuclear line through the centre of gravity of the molecule are equal. The presence of the electrons introduces a slight asymmetry, which gives rise to a fine doublet structure in the rotational levels (Λ-doubling) provided the moment of momentum Ω of the electrons around the internuclear line does not vanish (for π, Δ, ..., -states). Here, too, we shall get a Stark effect given by equation (5) whenever the electric moment r of the molecule is different from zero. If $\Omega = 0$ (Σ-states), there will be only an inappreciable Stark effect due to the terms of the perturbation matrix neglected in this section.

We can now give a brief summary of the cases in which an appreciable Stark effect may be expected when an electric field is applied to a radiating molecular system. For diatomic molecules in states with $\Omega \neq 0$ and a non-vanishing permanent moment there will be a splitting, quadratic at first and linear as soon as the energy shifts become large compared with the natural Λ-doubling intervals. A similar result holds for polyatomic molecules with two nearly equal moments of inertia. For polyatomic molecules with three appreciably different moments of inertia only the lowest rotational states will give a measurable splitting, quadratic in E.

I wish to express my thanks to the Board of Education and to the Kent Education Committee for a grant which made my stay in Groningen possible. I am also obliged to Professor Coster for permission to work in his institute, and to Dr. Kronig for many helpful suggestions during the course of this work.

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LI. *Measurements on the Degree of Orientation in Hard-drawn Copper Wires.* *By* W. A. Wood, *M.Sc., Physics Department, National Physical Laboratory, Teddington, Middlesex* *.

Introduction.

THE working of metals in the cold, in particular the hard-drawing of wires, tends to disturb the random arrangement of the crystalline grains in the material. In the case of a wire certain axes of the grains rotate themselves in some degree into alignment with the axis. The wire is then said to become oriented. A great amount of highly important work has resulted in the determination for a number of metals of those crystallographic axes which follow the line of the wire. Much less research has been done, however, with a view to estimating the proportion of the grains so oriented under different conditions of drawing.

The object of the present work was to determine how the degree of orientation varied across the section of hard-drawn copper wire, and also to follow the changes at the several stages occurring as the diameter of a wire was reduced by further cold drawing. The results obtained qualify to rather a large extent some previous work † on the subject, which states that the degree of orientation grows in a regular manner from the surface to the interior of the wire. Moreover, the X-ray photographs reveal information which it is thought may be of interest in connexion with the mechanical effects of dies and their comparative efficiency.

Experimental Procedure.

After the preliminary examination by X-rays of several specimens of commercial hard-drawn copper wire one characteristic sample, 1·80 mm. in diameter, was treated as follows:—Part of the specimen was annealed to remove all traces of orientation. The diameter of both pieces was then reduced by successive stages of drawing through the holes of a steel die plate until the final products had a diameter of about 0·5 mm. At thirty intervening steps

* Communicated by Dr. G. W. C. Kaye, O.B.E.

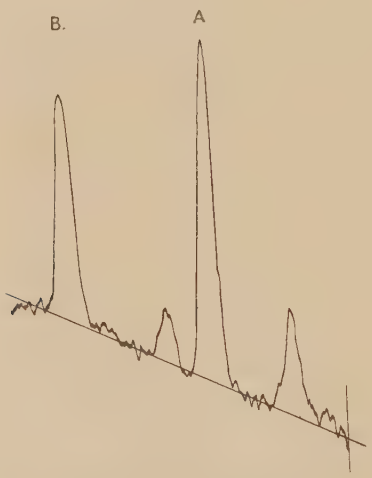
† *Cf.* Schmidt, *Zeit. für Metallkunde*, xx. p. 375 (1928).

in the drawing, referred to below as nos. 1 to 30, a sample some 15 cm. long was cut from the wire. Each sample obtained in this way was examined by the method of X-ray analysis. The apparatus used included a circular camera, 5 cm. in diameter and 10 cm. long, which fitted on a spectrometer table. The specimen could be rotated along the camera axis in a narrow beam of X-rays perpendicular to that axis. Copper Kα radiation was used for preliminary work, and then the more highly absorbed iron Kα for finer measurements. The procedure adopted then involved the taking of a Debye photograph of the surface of the wire. Next, the diameter of the wire was slightly reduced by electrolytically dissolving away the surface in dilute nitric acid. It was not a difficult matter to do this in a way which secured a uniform reduction along the whole wire. A photograph was then obtained of the new surface of the wire under the same conditions as before. Finally, by repeating the experiments one got together photographs showing the condition of the wire across the section of each of the various samples characterizing the different stages of the drawing.

The estimation of the degree of orientation was based upon a measurement of the density of the photographic films at convenient points on the diffraction rings where maxima or minima occurred as a result of the orientation. In the photograph of a non-oriented material the intensity at every point on a diffraction ring, omitting for the moment modifications produced by absorption and geometric considerations, should be the same. As a consequence of orientation maxima and minima of intensity appear on the ring. The ratio of the intensity at a minimum to that at a maximum on the same ring will therefore vary from zero for an ideally perfectly oriented specimen to the limiting value of unity for one showing no orientation at all. It is reasonable therefore to take the value of this ratio in measuring the degree of orientation in a given specimen and in comparing the degree of orientation obtaining in different specimens. In the present case it was considered more convenient to take, in the first place, the ratio of the intensity of the minimum associated with the [111] interference line to that of the maximum on the [200] line. The orientation of the copper is of the double fibre type, a cube diagonal [111] direction, and also a cube edge [100] direction,

tending to orient themselves parallel to the axis of the wire. The above points of measurement can therefore be chosen so as to be on that line on the photograph which is perpendicular to the length of the wire. This choice has the advantage that effects due to absorption are minimized and that an easier task is made of the photometering. The Moll self-recording microphotometer was used. The measurements on the records, taken in this way from the bromide paper that recorded a run on the instrument along the equatorial line passing through the above maximum and minimum, were next interpreted in

Fig. 1.



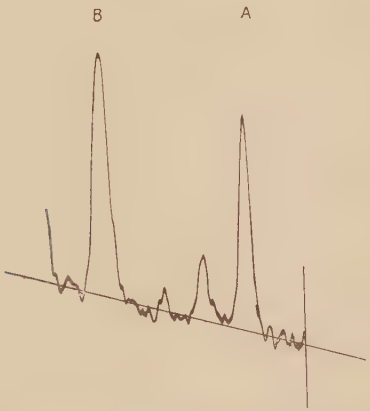
terms of actual photographic density by the aid of a calibration wedge of known density steps. The comparison of the area under the calibrated curves gave the ratio of minimum to maximum. Since, of course, the comparison is not now between maxima and minima on the same ring, this ratio will not tend to unity in the case of a non-oriented photograph. It will lead to the normal intensity ratio of the lines concerned. The reproduction of two records (figs. 1 and 2) illustrates the method. The [111] line occurs at the point A and the [200] at B. The ratio of A to B in fig. 1 is seen to be very different from the ratio A to B in fig. 2. The former is from the photograph of a non-oriented layer and the latter from an oriented layer of the same wire.

A second set of records was made comparing the intensity of the [111] minimum to that of the [202] maximum also on the equatorial line. These points form as a consequence of the same [111] orientation. It was found that the results described below were the same whichever set of records was considered. The two types of orientation, therefore, characteristic of the face-centred lattices, occurred in the same proportion in every photograph. It is interesting to note that this proportion is not affected by drawing.

The Core Effect.

The description of the results may be taken in two parts. In the first case we have the variation in the degree of

Fig. 2.

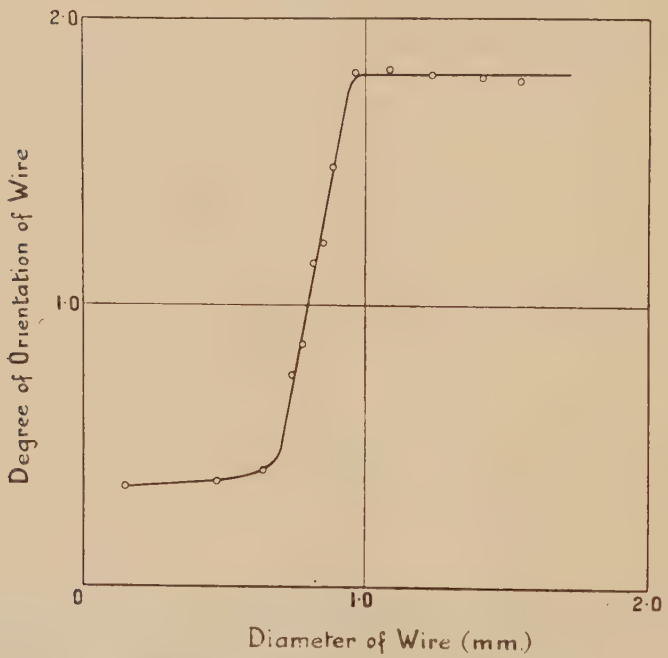


orientation across the diameter of each wire, and, secondly, the variation of that depth in the wires at which a given degree of orientation occurs as the wire is drawn. In the case of the parent specimen, which had received preliminary annealing, orientation occurred only after several degrees of drawing. From that stage on the results obtained were the same as those obtained from the specimen which was not annealed, and which showed, therefore, orientation from the commencement. Consequently the results from the latter extend over a wider range and are considered more closely.

The graph in fig. 3 shows the variation in degree of orientation across the section of the wire no. 3, the

original diameter of which was 1·76 mm. The ratio expressing the amount of orientation is plotted against the diameter of the wire. From the surface of the wire to a depth at which the diameter is equal to 0·95 mm. no sign of orientation was detected. The ratio A to B was constant and equal to about 1·8. Then, at a point when the diameter has a value of 0·90 mm., the ratio rapidly decreases, and in a short space assumes a value of about 0·4. This value is practically constant, though,

Fig. 3.



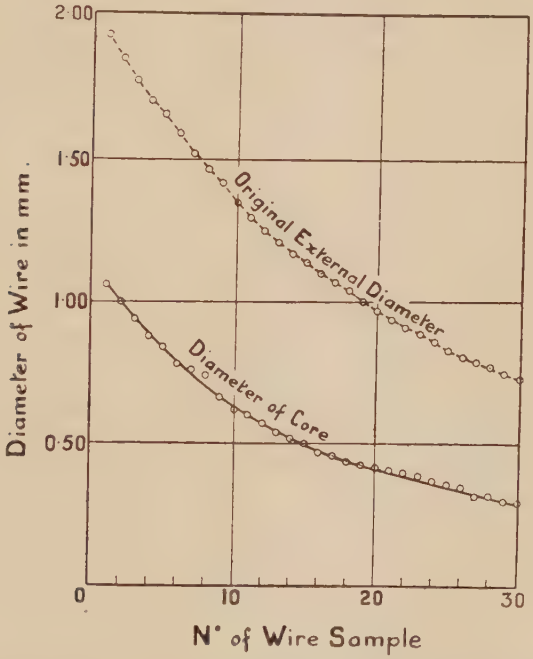
if anything, slightly decreasing as the centre of the wire is approached. The change from the non-oriented outer layers to the highly oriented inner ones is so sudden that one might describe the wire as consisting of a randomly oriented hollow cylinder surrounding a highly oriented core.

It was found, moreover, that the graph of wire no. 3 is a typical one. It represents equally well, if allowance be made for the scale of diameter, the variation across the section of each of the twenty-nine other wires. Moreover it indicated the internal condition also of many samples of

commercial hard-drawn wire obtained from other sources. It is surprising that the non-oriented layer occupies so large a fraction of an otherwise highly oriented wire.

The value of the diameter at the point at which the orientation just begins was next determined for each wire. In this way the variation in the diameter of the oriented core with further drawing could be seen. The results are given in fig. 4. The lower curve is obtained by plotting the value of this diameter against the number of the wire.

Fig. 4.



In the upper curve, on the same graph, the value of the original diameter of the wire is plotted against the number of the wire. The ratio of the ordinates of the lower to the upper curve for a given value of the abscissa then gives the fraction of any wire which is taken up by the oriented core. Thus, if the curves were parallel this fraction would be constant. It appears, firstly, from the smoothness of the graph that the diameter of the oriented core decreases uniformly as the outer diameter of the wire is decreased by drawing, and, secondly, that the curves

are nearly but not quite parallel. The proportion of the diameter occupied by the oriented core slightly decreases from 0·50 for wire no. 3, of external diameter 1·67 mm., to 0·41 for wire no. 30, of external diameter 0·73 mm. One concludes, therefore, that the proportionate amount to which the non-oriented boundary layers encroach upon the oriented core tends to increase as the diameter of the wire is reduced. On the other hand, it was found that the degree of orientation at the centre of the wire, as shown by interpolation of the graphs corresponding to fig. 3 for the various wires, slightly increased in the range of drawing under review. Consequently we have the result that whilst the layer of randomly oriented grains on the outer surface of the wire tends to extend more and more to the interior of the wire on drawing, yet the actual degree of orientation in the narrowing core of the wire tends to increase.

Interpretation of Results.

The results may be interpreted as follows :—The effect of drawing is to increase slowly the degree of orientation in the interior of a wire. This orientation is highest along the axis in a given wire. At the same time there exists another factor which causes a destruction of the orientation to a definitely marked depth in the wire. The change from the non-oriented portion to that oriented is quite sharp. This factor, it is suggested, is the mechanical effect of the edges of the holes in the die plate, and will characterize all drawn wire. The effect extends slightly more deeply as the wire is further drawn. The following experiment seems to confirm this suggestion. A wire was selected, the diameter equal to 1·67 mm., and from it the outer non-oriented layers were removed until the final surface just showed signs of orientation. It was known from the above work (*vide* fig. 3) that the slightest further reduction would give a surface highly oriented. The diameter was then 0·9 mm. At 0·83 mm. diameter, for instance, the orientation would be very marked. The wire was then drawn down to a diameter of 0·83 mm. Examined now with X-rays, after the manner described above, it was found that again the surface orientation disappeared. The wire had to be etched down to a diameter of 0·45 mm. before orientation was detected. At 0·40 mm. the orientation was again high. The action of the steel die would seem to be very speedy and very marked.

A similar type of investigation on rolled copper revealed the interesting fact that in this case the orientation began at once on the surface.

A common state of orientation in commercial hard-drawn copper wire seems to be described by the above results, and the use of X-rays in this province may be of practical value in comparing on the one hand the properties of different wires, and, on the other, in determining the efficiency of various methods of drawing. In works practice it is usually desirable to draw down a wire as far as possible before resorting to the process of annealing. An X-ray photograph of a sample, after further research on the above lines, might give an easy method of denoting the approach of the annealing point.

In conclusion, the author wishes to express his indebtedness to Mr. J. R. Clarkson, B.Sc., for help in much of the etching and photographic work, and also to Dr. G. Shearer for his constant interest in the research. The work is being extended to wires less drastically drawn

Summary.

1. An investigation is made of the degree of orientation across the section of hard-drawn copper wire and of the way these measurements vary as the wire is further drawn.

2. A method is described whereby the degree of orientation may be estimated with the aid of a photometer.

3. A core effect is found to exist. A non-oriented layer encloses an oriented core and the line of demarcation is quite sharp. The results are discussed in connexion with the mechanical action of dies.

September 1930.

LII. *The Stability of Motion of Rectilinear Vortices in Ring Formation.* *By* T. H. Havelock, *F.R.S.**

Introduction and Summary.

1. THE stability of the two-dimensional motion of an infinite system of vortices arranged in a single row, or in double rows has been worked out in detail in recent years, but not much attention has been given to cases in

* Communicated by the Author.

which the number of vortices is finite. The obvious analogous problems arise when the vortices are equally spaced round the circumference of one or more concentric rings; the problems are not perhaps of special importance, but they are of some interest, and, further, one may obtain the infinite straight rows as limiting cases of ring formation.

We examine first the motion of a single ring of vortices, a problem which attracted attention many years ago in connexion with the vortex theory of atoms. Kelvin* worked out the case of three vortices, but failed to obtain a solution for a larger number; it was in this connexion that he drew attention to the now well-known experiments of Mayer with floating magnets. Shortly afterwards the problem was attacked by J. J. Thomson †, and it is usually stated that he proved the configuration to be stable if, and only if, the number of vortices does not exceed six. He, in fact, worked out the small oscillations for the particular cases of three, four, five, six, and seven vortices, obtaining an instability in the last case. It appears that the equations for the general case are capable of a simple explicit solution, and this is given in § 2; a ring of seven vortices is neutral for small displacements, with less than seven it is completely stable, and for more than seven unstable. In § 3 the effect of an assigned velocity field in addition to that of the vortices is examined briefly.

In the next two sections we work out the effect of a concentric circular boundary upon the stability of a single ring, the boundary being either internal or external to the ring. In both cases the stability is diminished, seven or more vortices being unstable whatever the radius of the boundary. For a smaller number there is a limiting ratio of the radius of the ring to the radius of the boundary for stability in each case. For an external boundary the motion is unstable in any case if the radius of the boundary is less than about twice the radius of the ring, and there is a similar result for an internal boundary. The effect of the boundary, estimated in this way, seems larger than might have been anticipated.

In the remaining sections we examine the motion of two concentric rings of vortices, of opposite rotations, the vortices being spaced alternately. A steady state is possible in which the rings rotate and retain their relative positions unaltered, but there are always modes of disturbance which give rise

* Kelvin, Math. and Phys. Papers, iv. p. 135 (1878).
† J. J. Thomson, 'Treatise on Vortex Rings,' p. 94 (1883).

to instability. By suitable choice of the relative strengths of the vortices in the two rings it is possible to limit the instability to only one special mode of disturbance; it is this particular configuration which becomes in the limit the stable Karman vortex street, when we make the radius of a ring and the number of vortices both infinite, keeping their ratio finite.

Single Ring of Vortices.

2. Let there be n equal vortices, each of strength κ, equally spaced round the circumference of a ring of radius a. In steady motion the ring rotates with a certain angular velocity ω. Let the vortices be slightly displaced, and suppose the disturbed positions to be given in polar co-ordinates by

$$a+r_{s+1}, \quad 2s\pi/n+\omega t+\theta_{s+1}, \quad \ldots \quad (1)$$

where $s=0, 1, \ldots n-1$, and r, θ are small radial and angular displacements from the steady state. Consider the motion of one of the vortices, say that at the point $(a+r_1, \omega t+\theta_1)$; its velocity is due to the other vortices, and the radial component is

$$\frac{\kappa}{2\pi}\sum_{s=1}^{n-1}\frac{(a+r_{s+1})\sin(2s\pi/n+\theta_{s+1}-\theta_1)}{D^2}, \quad \ldots \quad (2)$$

while the transverse component is

$$-\frac{\kappa}{2\pi}\sum_{s=1}^{n-1}\frac{(a+r_{s+1})\cos(2s\pi/n+\theta_{s+1}-\theta_1)-(a+r_1)}{D^2}, \quad (3)$$

where

$$D^2=(a+r_{s+1})^2+(a+r_1)^2 -2(a+r_{s+1})(a+r_1)\cos(2s\pi/n+\theta_{s+1}-\theta_1).$$

We expand these expressions to the first order terms in r and θ, and so get the equations of motion of the vortex under consideration. After some reduction we obtain

$$\dot{r}_1=-\frac{\kappa}{4\pi a}\sum_1^{n-1}\frac{\theta_{s+1}-\theta_1}{1-C_s},$$

$$(a+r_1)\omega+a\dot{\theta}_1=\frac{\kappa}{4\pi a}\sum_1^{n-1}\left\{1+\frac{C_s}{1-C_s}\frac{r_1}{a}-\frac{1}{1-C_s}\frac{r_{s+1}}{a}\right\}, \quad (4)$$

where $C_s=\cos(2s\pi/n)$.

The steady state is given by $\omega=(n-1)\kappa/4\pi a^2$, and since

$$\sum_1^{n-1}\frac{1}{1-C_s}=\frac{1}{6}(n^2-1),$$

equations (4) give

$$(4\pi a/\kappa)\dot{r}_1 = A\theta_1 - \sum_1^{n-1} a_s\theta_{s+1},$$

$$(4\pi a^3/\kappa)\dot{\theta}_1 = Br_1 - \sum_1^{n-1} a_s r_{s+1}, \quad . \quad . \quad . \quad . \quad (5)$$

where

$$A = \frac{1}{6}(n^2-1); \quad B = \frac{1}{6}(n-1)(n-11); \quad a_s = 1/(1-C_s).$$

There are similar equations for each vortex, giving altogether a system of $2n$ equations.

The simplest method of treating the equations is to examine a possible simple solution of the form

$$r_{s+1} = \alpha e^{2ks\pi i/n}; \quad \theta_{s+1}\beta e^{2ks\pi i/n}, \quad . \quad . \quad . \quad . \quad (6)$$

where $k = 0, 1, 2, \ldots n-1$.

It may be proved that under the conditions stated

$$\sum_{s=1}^{n-1} \frac{e^{2ks\pi i/n}}{1-\cos(2s\pi/n)} = \frac{1}{6}(n^2-1) - k(n-k) \quad . \quad . \quad (7)$$

Hence, from (5), we find that the equations for α, β reduce to

$$(4\pi a/\kappa)\dot{\alpha} = k(n-k)\beta,$$

$$(4\pi a^3/\kappa)\dot{\beta} = \{k(n-k) - 2(n-1)\}\alpha. \quad . \quad . \quad . \quad (8)$$

Finally, taking α and β to be proportional to $e^{\lambda t}$, these give

$$\lambda^2 = \left(\frac{\kappa}{4\pi a^2}\right)^2 k(n-k)\{k(n-k) - 2(n-1)\} \quad . \quad . \quad (9)$$

It follows that in (6), (8), and (9) we have, in general, $2n$ independent solutions of the equations of the system, and that we can build up the complete solution for any arbitrary small initial displacements of the vortices.

An alternative method of solution may be noticed briefly, namely, the method used by previous writers for particular cases; it may be extended to give the general results, though not quite so simply as in (6)–(9). In the $2n$ equations (5) we assume each coordinate to be proportional to $e^{\lambda t}$, and form the determinantal equation for λ. The determinant can be reduced to one of order n in λ^2, and it can be shown that it is of the type known as a circulant, and can be factorized in terms of the nth roots of unity; after some reduction we obtain (9) again, and can deduce the corresponding simple solutions given by (6).

From (9), when $k=0$ we have $\lambda=0$. If we examine this case we find that the displacement consists of a rotation of the ring combined with a small change in its radius ; the result is a new steady state with a corresponding small change in the angular velocity. The condition for stability is that λ^2 must be negative for all the other values of k, namely, 1, 2, ... $n-1$. Hence, from (9), the steady state is stable if

$$k(n-k)-2(n-1) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

is negative for all the values of k, and this is the case if it is negative for $k=\frac{1}{2}n$ when n is even, or $\frac{1}{2}(n\pm1)$ when n is odd. It follows at once that the steady state is completely stable when $n<7$. When $n=7$ the expression (10) is zero for $k=3$ or 4 ; while for $n>7$ there are always some values of k for which λ^2 is positive, and hence the system is unstable.

Whatever the value of n there are always two modes of possible small oscillations, namely, those given by $k=1$ and $k=2$.

When $k=2$ we have

$$\lambda^2=-4\left(\frac{\kappa}{4\pi a^2}\right)^2(n-2), \quad . \quad . \quad . \quad . \quad (11)$$

while for $k=1$

$$\lambda^2=-\left(\frac{\kappa}{4\pi a^2}\right)^2(n-1)^2=-\omega^2. \quad . \quad . \quad . \quad (12)$$

We notice that in the latter case the period of the small oscillation is the same as the period of rotation of the ring in the steady state ; this motion was worked out for the particular case of three vortices by Kelvin in the paper already quoted, and it is illustrated in a characteristic manner by the description of a working model to show the motion of the vortices.

The single infinite straight row of vortices may be obtained by making both n and a become infinite, with the ratio $n/2\pi a$ finite and becoming in the limit equal to the distance between consecutive vortices ; the usual results then follow from (6) and (9).

Single Ring in assigned Field.

3. We have so far considered the vortices to be moving solely under their mutual actions. Suppose now that there is an assigned velocity field which is maintained independently ; for simplicity we suppose the flow to be in circles

round the origin, the angular velocity being $\Omega(r)$, and the transverse fluid velocity at a distance r being $r\Omega$.

Then, referring to equations (4) for the motion of a typical vortex in the ring, the only difference is that we have to add

$$a\Omega(a)+r_1\{\Omega(a)+a\Omega'(a)\}$$

on the right-hand side of the second equation. The angular velocity of the ring in the steady state is now

$$\cdot\ \cdot\ \cdot(n-1)\kappa/4\pi a^2+\Omega(a).$$

Following the same procedure, we obtain, instead of (8), the equations

$$(4\pi a/\kappa)\dot{\alpha}=k(n-k)\beta,$$

$$(4\pi a^3/\kappa)\dot{\beta}=\{k(n-k)-2(n-1)+(4\pi a^3/\kappa)\Omega'(a)\}, \quad (13)$$

and hence we have

$$\lambda^2=k(n-k)\{k(n-k)-2(n-1)+(4\pi a^3/\kappa)\Omega'(a)\}, \quad (14)$$

with $k=0, 1, \ldots n-1$.

It follows that the steady state can be stabilized for any value of n, provided $\Omega'(a)$ is negative and sufficiently large.

Two special cases may be noted. First, if the fluid is rotating like a rigid body—that is, if $\Omega(r)$ is constant—the conditions for stability are unaffected. In the second place, suppose there is an assigned vortex fixed at the origin, so that $\Omega(r)=\kappa'/2\pi r^2$; then, if κ' is of the same sign as κ, we can make the steady state stable for any value of n by taking κ' large enough.

Single Ring with Outer Boundary.

4. Suppose the liquid is contained within a circular boundary of radius b, the vortices being in the steady state on a concentric circle of radius a ($<b$). The motion in the liquid is due to the given vortices and their images in the circular boundary.

Taking the steady state first, the radius of the image ring is b^2/a, the strength of each image vortex being $-\kappa$. Writing down the velocity at any vortex in the given ring, the angular velocity in the steady state is given by

$$a\omega=\frac{(n-1)\kappa}{4\pi a}+\frac{\kappa}{2\pi}\sum_{s=0}^{n-1}\frac{(b^2/a)\mathrm{C}-a}{b^4/a^2+a^2-2b^2\mathrm{C}}, \quad (15)$$

where $\mathrm{C}=\cos(2s\pi/n)$.

We shall have occasion to use the following summations, which can easily be proved :

$$\sum_{s=1}^{n-1}\frac{1-p^2}{1-2pC+p^2}=\frac{2n}{1-p^n}-\frac{2}{1-p}-(n-1),$$

$$\sum_{s=1}^{n-1}\frac{1-pC}{1-2pC+p^2}=\frac{n}{1-p^n}-\frac{1}{1-p},$$

$$\sum_{s=1}^{n-1}\frac{(1+p^2)(1-2pC+p^2)}{(1-2pC+p^2)^2}=\frac{n^2p^{n-1}}{(1-p^n)^2}-\frac{1}{(1-p)^2}, \quad . \quad . \quad (16)$$

with $0<p<1$.

Writing $p=a^2/b^2$, we find from (15)

$$a\omega=\frac{\kappa}{4\pi a}\left(\frac{2n}{1-p^n}-n-1\right). \quad . \quad . \quad . \quad (17)$$

For small displacements from the steady state we have for each vortex κ at a point

$$a+r_{s+1}, \quad \omega t+2s\pi/n+\theta_{s+1},$$

an image vortex $-\kappa$ at the point

$$\frac{b^2}{a}\left(1-\frac{r_{s+1}}{a}\right), \quad \omega t+2s\pi/n+\theta_{s+1}.$$

Considering the motion of the vortex given by $s=0$, we have for the radial velocity the expression (2), together with

$$-\frac{\kappa}{2\pi}\sum_{s=0}^{n-1}\frac{(b^2/a)(1-r_{s+1}/a)\sin\phi}{E^2}, \quad . \quad . \quad (18)$$

and for the transverse velocity we have (3), together with

$$\frac{\kappa}{2\pi}\sum_{s=0}^{n-1}\frac{(b^2/a)(1-r_{s+1}/a)\cos\phi-(a+r_1)}{E^2}, \quad . \quad (19)$$

where

$$\phi=2s\pi/n+\theta_{s+1}-\theta_1,$$

and

$$E^2=\frac{b^4}{a^2}\left(1-\frac{r_{s+1}}{a}\right)^2+(a+r_1)^2-2\frac{b^2}{a}\left(1-\frac{r_{s+1}}{a}\right)(a+r_1)\cos\phi.$$

The steps in the reduction of the equations of motion need not be reproduced here ; making use of the summations given in (16), and writing

$$p=a^2/b^2 ; \quad S=\sin(2s\pi/n) ; \quad C=\cos(2s\pi/n), \quad . \quad (20)$$

we obtain eventually the equations

$$\frac{4\pi a^2}{\kappa}\frac{\dot{r}_1}{a} = \left\{\frac{n^2-1}{6} + \frac{2n^2p^n}{(1-p^n)^2} - \frac{2p}{(1-p)^2}\right\}\theta_1$$
$$- \sum_{s=1}^{n-1}\left\{\frac{1}{1-C} + \frac{2p\{(1+p^2)C-2p\}}{(1-2pC+p^2)^2}\right\}\theta_{s+1}$$
$$- \sum_{s=1}^{n-1}\frac{2p(1-p^2)S}{(1-2pC+p^2)^2}\frac{r_{s+1}}{a}$$

$$\frac{4\pi a^2}{\kappa}\dot{\theta}_1 = \left\{\frac{n^2-1}{6} + 2(n-1) + \frac{2n^2p^n}{(1-p^n)^2}\right.$$
$$\left. - \frac{4n}{1-p^n} + \frac{2(2p^2-3p+2)}{(1-p)^2}\right\}\frac{r_1}{a}$$
$$- \sum_{s=1}^{n-1}\left\{\frac{1}{1-C} - \frac{2p\{(1+p^2)C-2p\}}{(1-2pC+p^2)^2}\right\}\frac{r_{s+1}}{a}$$
$$- \sum_{s=1}^{n-1}\frac{2p(1-p^2)S}{(1-2pC+p^2)^2}\theta_{s+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

There are $2n$ equations of this type, and we examine now a possible solution of the form

$$r_{s+1}/a = \alpha e^{2ks\pi i/n}; \quad \theta_{s+1} = \beta e^{2ks\pi i/n}, \quad . \quad . \quad . \quad (22)$$

with $k = 0, 1, \ldots, n-1$.

On substituting these expressions we obtain two equations in α and β. In simplifying the various coefficients we use the following summations, whose proof need not be given here:—

$$\sum_{s=1}^{n-1}\frac{(1-p^2)\cos(2ks\pi/n)}{1-2pC+p^2} = \frac{n(p^k+p^{n-k})}{1-p^n} - \frac{2}{1-p} + 1$$
$$\sum_{1}^{n-1}\frac{(1-pC)\cos(2ks\pi/n)}{1-2pC+p^2} = \frac{n(p^k+p^{n-k})}{2(1-p^n)} - \frac{1}{1-p}$$
$$\sum_{1}^{n-1}\frac{\{(1+p^2)C-2p\}\cos(2ks\pi/n)}{(1-2pC+p^2)^2} = \frac{nk(p^{k-1}-p^{n-k-1})}{2(1-p^n)}$$
$$+ \frac{n^2p^{n-1}(p^k-p^{-k})}{2(1-p^n)^2} - \frac{1}{(1-p)^2}, \quad . \quad . \quad (23)$$

valid for $0 < p < 1$, and $k = 1, 2, \ldots, n-1$.

We obtain after some reduction the equations

$$(4\pi a^2/\kappa)\dot{\alpha} = P\beta - iR\alpha,$$
$$(4\pi a^2/\kappa)\beta = Q\alpha - iR\beta, \quad . \quad . \quad . \quad (24)$$

where

$$P=k(n-k)-\frac{nk(p^k-p^{n-k})}{1-p^n}-\frac{n^2p^{n-k}(1-p^k)^2}{(1-p^n)^2},$$

$$Q=k(n-k)+2(n+1)+\frac{nk(p^k-p^{n-k})}{1-p^n}+\frac{n^2p^{n-k}(1+p^k)^2}{(1-p^n)^2},$$

$$R=\frac{nk(p^k+p^{n-k})}{1-p^n}-\frac{n^2p^{n-k}(1-p^{2k})}{(1-p^n)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

We may check these expressions by deducing the equations for the corresponding disturbance of an infinite double symmetrical row of vortices. If d is the distance between consecutive vortices in each row, and h the distance between the two rows, we have, in the limit,

$$2\pi a/n=d\,; \quad 2k\pi/n=\phi\,;$$
$$p=(1+2\pi h/nd)^{-1}.$$

With these (17) gives the limiting value of the linear velocity of the vortices, namely,

$$\frac{\kappa}{2d}\coth\frac{\pi h}{d};$$

further, the quantities $n^2P/2\pi^2$, $n^2Q/2\pi^2$, and $n^2R/2\pi^2$ become respectively the quantities A+C, A−C, and B in the notation of Lamb's 'Hydrodynamics,' (5th ed. p. 221).

Returning to equations (24), we take α and β proportional to

$$e^{\kappa\lambda t/4\pi a^2},$$

and obtain

$$\lambda=-iR\pm(PQ)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

For complete stability the product PQ must be negative, or zero possibly, for all the values of k. To prove instability it is sufficient to show that PQ is positive for one value at least of k. From the form of the expressions in (25) we see that P and Q are symmetrical in k and $n-k$, and that the critical mode to examine is $k=\frac{1}{2}n$ for n even, or $k=\frac{1}{2}(n\pm1)$ for n odd.

For n even we have

$$P(\tfrac{1}{2}n)=\tfrac{1}{4}n^2-n^2p^{\frac{1}{2}n}/(1+p^{\frac{1}{2}n})^2, \quad . \quad . \quad . \quad (27)$$

which is always positive. Further,

$$Q(\tfrac{1}{2}n)=\tfrac{1}{2}n^2+2(n+1)-\frac{4n}{1-p^n}+\frac{n^2p^{\frac{1}{2}n}}{(1-p^{\frac{1}{2}n})^2}, \quad . \quad (28)$$

and this is positive if

$$(n^2+8n+8)x^3+(3n^2-8n-8)x^2+(3n^2+8n-8)x+n^2-8n+8>0, \quad . \quad (29)$$

where $x=p^{\frac{1}{2}n}=(a/b)^n$.

The left-hand side of (29) is always positive for $n\geqslant 8$. From similar expressions when n is odd we find that there is always a positive value of Q for $n=7$. Hence we conclude that the motion of the ring is unstable when the number of vortices is equal to or greater than seven, whatever the radius of the outer boundary. For $n<7$ we shall see that the motion is stable provided the ratio of the radius of the ring to that of the boundary is less than a certain value in each case. We shall examine the cases briefly, noting that in each case the mode $k=0$ means simply a neutral displacement of the ring.

For $n=2$, $k=1$, we find from the previous expressions that Q is negative if $p<0{\cdot}2137$; and as P is positive, it follows that the circular motion of the two vortices is stable if $a/b<0{\cdot}462$.

For $n=3$, $k=1$ or 2, Q is negative for $p<0{\cdot}322$, and the motion is stable for $a/b<0{\cdot}567$.

Similarly for $n=4$ we find the critical value of a/b to be about $0{\cdot}575$; for $n=5$ it is $0{\cdot}588$, and for $n=6$ it is $0{\cdot}547$. When $n=7$, which is the critical neutral case when there is no boundary, the effect of an outer boundary of any radius is to cause instability.

Single Ring with Inner Boundary.

5. Suppose now that the fluid is bounded internally by a circular barrier ($r=b$), and that a ring of n vortices is rotating in circular motion in a ring of radius a ($>b$). The image of a vortex κ at $r=a$ is a vortex $-\kappa$ at $r=b^2/a$, together with a vortex κ at $r=0$; this combination makes the circulation zero for a circuit enclosing the boundary without including any of the actual vortices.

We find the equations of motion of a given vortex, $s=0$ in the previous notation, just as in § 4. The only differences arise (i.) from the additional image vortex $n\kappa$ at the origin, and (ii.) in evaluating the various summations, as b/a is now less than unity instead of a/b. For the steady state we have

$$a\omega=\frac{(n-1)\kappa}{4\pi a}+\frac{n\kappa}{2\pi a}-\frac{\kappa}{2\pi a}\sum_{s=0}^{n-1}\frac{1-q\mathrm{C}}{1-2q\mathrm{C}+q^2}, \quad . \quad (30)$$

where $q=b^2/a^2$ and $\mathrm{C}=\cos(2s\pi/n)$.

This gives

$$a\omega = \frac{\kappa}{4\pi a}\left(3n-1-\frac{2n}{1-q^n}\right). \quad . \quad . \quad . \quad (31)$$

We shall merely state now the results for the general equations of disturbed motion. The equations for $\dot{r}_1$ and $\dot{\theta}_1$ are the same as in (21), with the following alterations:—(i.) write q for p in the coefficients, (ii.) change the sign of the last term in each equation, the coefficients of r_{s+1} and θ_{s+1} respectively, from $-$ to $+$, (iii.) change the coefficient of r_1 in the second equation to

$$\frac{1}{6}(n^2-1)-2(n-1)+\frac{4nq^n}{1-q^n}+\frac{2n^2q^n}{(1-q^n)^2}+\frac{2q}{(1-q)^2}.$$

Taking a simple solution of type (22), and proceeding as in (24), (25), we obtain, instead of (26), the result

$$\lambda = iR' \pm (P'Q')^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where

$$P' = k(n-k)-\frac{nk(q^k-q^{n-k})}{1-q^n}-\frac{n^2q^{n-k}(1-q^k)^2}{(1-q^n)^2},$$

$$Q' = k(n-k)-2(n-1)+\frac{4nq^n}{1-q^n}+\frac{n^2q^{n-k}(1-q^k)^2}{(1-q^n)^2}$$

$$+\frac{(nk(q^k-q^{n-k})}{1-q^n},$$

$$R' = \frac{nk(q^k+q^{n-k})}{1-q^n}-\frac{n^2q^{n-k}(1-q^{2k})}{(1-q^n)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

As before, it appears that stability depends upon there being values of q less than unity for which Q' is negative for all the values of k. It is easily seen that there is no such value of q when $n \geq 7$, and therefore the steady state is unstable when there are seven or more vortices in the ring.

Examining the expressions numerically for smaller values of n, we find that the steady state is stable under the following conditions:—$n=2$, $b/a<0{\cdot}386$; $n=3$, $b/a<0{\cdot}522$; $n=4$, $b/a<0{\cdot}556$; $n=5$, $b/a<0{\cdot}579$; $n=6$, $b/a<0{\cdot}544$.

These values are slightly less than the corresponding limits when the ring is within the circular boundary, but there is little difference in the general conclusions.

Double Alternate Rings.

6. In the previous sections we have been considering in effect a double symmetrical ring, in which the motions of one ring—the image ring—are constrained in accordance with those of the actual ring. We shall leave on one side the general case of a free double symmetrical ring, and proceed to two alternate rings in an unlimited liquid.

Let there be n positive vortices, each of strength κ, equally spaced round a circle of radius a, and n negative vortices of strength κ' equally spaced round a concentric circle of radius b $(>a)$, the arrangement of the vortices being alternate. Thus, if the vortices in the inner ring are given by polar coordinates a, $2s\pi/n$, those of the outer ring are given by b, $2(s+\frac{1}{2})\pi/n$, with $s=0, 1, \ldots, n-1$.

Examine first the possibility of a steady state with the two rings rotating with equal angular velocity, the relative configuration remaining unchanged. The radial velocity of any vortex is zero. The transverse velocity of a vortex in the inner ring is given by

$$\frac{(n-1)\kappa}{4\pi a}+\frac{\kappa'}{2\pi}\sum_{s=0}^{n-1}\frac{bC'-a}{b^2+a^2-2abC'}, \quad . \quad . \quad (34)$$

and in the outer ring by

$$-\frac{(n-1)\kappa'}{4\pi b}-\frac{\kappa}{2\pi}\sum_{s=0}^{n-1}\frac{aC'-b}{a^2+b^2-2abC'}, \quad . \quad . \quad (35)$$

where $C'=\cos\{2(s+\frac{1}{2})\pi/n\}$.

We shall require the following summations, with $p=a/b<1$:

$$\sum_{s=0}^{n-1}\frac{1-pC'}{1-2pC'+p^2}=\frac{n}{1+p^n},$$

$$\sum_{0}^{n-1}\frac{1-p^2}{1-2pC'+p^2}=\frac{n(1-p^n)}{1+p^n},$$

$$\sum_{0}^{n-1}\frac{(1+p^2)C'-2p}{(1-2pC'+p^2)^2}=-\frac{n^2p^{n-1}}{(1+p^n)^2}. \quad . \quad . \quad (36)$$

The condition for equal angular velocity of the two rings then becomes

$$(n-1)\kappa-\frac{2np^n}{1+p^n}\kappa'=\frac{2np^2}{1+p^n}\kappa-(n-1)p^2\kappa'. \quad . \quad (37)$$

It can be seen that for a given ratio of κ' to κ we obtain from this equation a corresponding value of p less than unity,

and hence a possible steady state. Consider now the general equations for the disturbed motion. Let the positions of the vortices in the inner ring be given by polar coordinates

$$a(1+r_{s+1}), \quad 2s\pi/n+\omega t+\theta_{s+1},$$

and those in the outer ring by

$$b(1+\rho_{s+1}), \quad 2(s+\tfrac{1}{2})\pi/n+\omega t+\phi_{s+1}.$$

We form the equations of motion as in the previous sections. We choose a typical vortex, $s=0$, in the inner ring, and to simplify the notation we take the vortex $s=n-1$ in the outer ring. Expanding the components of velocity to first order terms, and reducing the coefficients by means of (36), we obtain the equations for these two vortices :

$$\begin{aligned}
4\pi a^2\dot{r}_1 = \kappa\Big\{&\frac{1}{6}(n^2-1)-\frac{2n^2p^n}{(1+p^n)^2}\Big\}\theta_1 \\
&-\kappa\sum_1^{n-1}\frac{\theta_{s+1}}{1-C}-\kappa'\sum_0^{n-1}\frac{2p\{(1+p^2)C'-2p\}}{D^2}\phi_{s+1} \\
&+\kappa'\sum_0^{n-1}\frac{2p(1-p^2)S'}{D^2}\rho_{s+1},
\end{aligned}$$

$$\begin{aligned}
4\pi a^2\dot{\theta}_1 = \kappa\Big\{&\frac{1}{6}(n^2-1)-2(n-1)+\frac{4np^n}{1+p^n}-\frac{2n^2p^n}{(1+p^n)^2}\Big\}r_1 \\
&-\kappa\sum_1^{n-1}\frac{r_{s+1}}{1-C}-\kappa'\sum_0^{n-1}\frac{2p\{(1+p^2)C'-2p\}}{D^2}\rho_{s+1} \\
&-\kappa'\sum_0^{n-1}\frac{2p(1-p^2)S'}{D^2}\phi_{s+1},
\end{aligned}$$

$$\begin{aligned}
4\pi b^2\dot{\rho}_n = -\kappa'\Big\{&\frac{1}{6}(n^2-1)-\frac{2n^2p^n}{(1+p^n)^2}\Big\}\phi_n \\
&+\kappa'\sum_1^{n-1}\frac{\phi_{s+1}}{1-C}+\kappa\sum_0^{n-1}\frac{2p\{(1+p^2)C'-2p\}}{D^2}\theta_{s+1} \\
&+\kappa\sum_0^{n-1}\frac{2p(1-p^2)S'}{D^2}r_{s+1}
\end{aligned}$$

$$\begin{aligned}
4\pi b^2\dot{\phi}_n = -\kappa'\Big\{&\frac{1}{6}(n^2-1)-2(n-1)+\frac{4n}{1+p^n}-\frac{2n^2p^n}{(1+p^n)^2}\Big\}\rho_n \\
&+\kappa'\sum_1^{n-1}\frac{\rho_{s+1}}{1-C}+\kappa\sum_0^{n-1}\frac{2p\{(1+p^2)C'-2p\}}{D^2}r_{s+1} \\
&-\kappa\sum_0^{n-1}\frac{2p(1-p^2)S'}{D^2}\theta_{s+1}, \qquad \ldots\ldots \quad (38)
\end{aligned}$$

where

$$C=\cos(2s\pi/n);\quad C'=\cos\{2(s+\tfrac{1}{2})\pi/n\};$$
$$S'=\sin\{2(s+\tfrac{1}{2})\pi/n\};\quad D=1-2pC'+p^2.$$

We now assume a simple solution of the form

$$r_{s+1}=\alpha e^{2ks\pi i/n};\qquad \theta_{s+1}=\beta e^{2ks\pi i/n};$$
$$\rho_{s+1}=\alpha' e^{2k(s+\frac{1}{2})\pi i/n};\quad \phi_{s+1}=\beta' e^{2k(s+\frac{1}{2})\pi i/n};\quad . \quad (39)$$

and, further, suppose that α, β, α', β' involve the time as a factor

$$e^{\kappa\lambda t/4\pi a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

In simplifying the coefficients we use the following summations, valid for $k=1, 2, \ldots, n-1$, which may be proved without difficulty:

$$\sum_0^{n-1}\frac{(1-p^2)E}{1-2pC'+p^2}=\frac{n(p^k-p^{n-k})}{1+p^n},$$

$$\sum_0^{n-1}\frac{2p\{(1+p^2)C'-2p\}E}{(1-2pC'+p^2)^2}=\frac{n\{kp^k-(n-k)p^{n-k}\}}{1+p^n}-\frac{n^2p^n(p^k-p^{n-k})}{(1+p^n)^2},$$

$$\sum_0^{n-1}\frac{2p(1-p^2)S'E}{(1-2pC'+p^2)^2}=i\left\{\frac{nk(p^k-p^{n-k})}{1+p^n}-\frac{n^2p^n(p^k-p^{-k})}{(1+p^n)^2}\right\}, \quad . \quad (41)$$

where $E=e^{2k(s+\frac{1}{2})\pi i/n}$.

The $4n$ equations of the system now reduce to

$$\lambda\alpha=P_1\beta+Q'\alpha'+R'\beta',$$
$$\lambda\beta=P_2\alpha+R'\alpha'-Q'\beta',$$
$$\lambda\alpha'=P_1'\alpha'+Q\alpha+R\beta,$$
$$\lambda\beta'=P_2'\alpha'+R\alpha-Q\beta, \quad . \quad . \quad . \quad . \quad . \quad (42)$$

where

$$\kappa P_1=-\frac{2n^2p^n}{(1+p^n)^2}\kappa'+k(n-k)\kappa,$$

$$\kappa P_1'=\frac{2n^2p^{n+2}}{(1+p^n)^2}\kappa-k(n-k)p^2\kappa',$$

$$\kappa P_2=\{k(n-k)-2(n-1)\}\kappa+2\left\{\frac{2np^n}{1+p^n}-\frac{n^2p^n}{(1+p^n)^2}\right\}\kappa',$$

$$\kappa P_2' = -\{k(n-k)-2(n-1)\}p^2\kappa' - 2\left\{\frac{2n}{1+p^n} - \frac{n^2p^n}{(1+p^n)^2}\right\}p^2\kappa,$$

$$Q = ip^2n\,\frac{p^k\{k-(n-k)p^n\}+p^{n-k}(n-k-kp^n)}{(1+p^n)^2},$$

$$R = p^2n\,\frac{p^k\{k-(n-k)p^n\}-p^{n-k}(n-k-kp^n)}{(1+p^n)^2},$$

$$Q' = \kappa' Q/\kappa p^2\,; \quad R' = -\kappa' R/\kappa p^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

The equation for λ is

$$\begin{vmatrix} \lambda & -P_1 & -Q' & -R' \\ -P_2 & \lambda & -R' & Q' \\ -Q & -R & \lambda & -P_1' \\ -R & Q & -P_2' & \lambda \end{vmatrix} = 0. \quad (44)$$

Using the relation (37) we see that $P_1 - P_2 = P_1' - P_2'$, and we find that (44) reduces to a quadratic in λ^2, which can be solved in the form

$$4\lambda^2 = (L^{\frac{1}{2}} \pm M^{\frac{1}{2}})^2 - (P_1 - P_2)^2, \quad . \quad . \quad . \quad (45)$$

where

$$L = (P_1 + P_2')^2 + 4QQ',$$

$$M = (P_1 - P_1')^2 + 4RR'. \quad . \quad . \quad . \quad . \quad (46)$$

The condition for complete stability is that for the n values of k all the values of λ^2 must be real and negative, including possibly zero. From the symmetry of the coefficients (43) in k and $n-k$ it is only necessary to examine the values of k from zero up to $\frac{1}{2}n$ if n is even, or $\frac{1}{2}(n-1)$ if n is odd.

7. We might examine now in detail the case when we take $\kappa' = \kappa$, that is, when the vortices in the two rings are of equal strengths and opposite rotations; we shall state the results without giving the details of the algebraic analysis. It can be shown that when p satisfies the equation (37) the quantity QQ' increases in absolute value from $k=0$ to $k=\frac{1}{2}n$, while the quantity RR' decreases in absolute value as k increases in this range. Further, except at $k=0$ the quantity L of (46) is always negative, and thus the criterion for stability is reduced to M being negative. But when $k=\frac{1}{2}n$ we have $R=R'=0$; and since P_1 is not in general equal to P_1', it follows that M is positive at $k=\frac{1}{2}n$. Hence, if n is even, the system is unstable for the mode $k=\frac{1}{2}n$ at least. It can be seen that in general there are always some unstable modes in the neighbourhood of this mean value of k;

further, there is always an instability associated with the mode $k=0$.

8. It remains to be seen whether we can obtain a greater degree of stability by a suitable choice of the ratio of κ' to κ, that is, with vortices of different strengths in the two rings. To avoid complicating the discussion we shall assume n even. Then the previous discussion suggests that we make the central mode stable, that is, we fix κ' by the condition that $P_1=P_1'$ at $k=\frac{1}{2}n$. From (43) this gives

$$\tfrac{1}{4}\kappa-\frac{2p^n}{(1+p^n)^2}\kappa'=\frac{2p^{n+2}}{(1+p^n)^2}\kappa-\tfrac{1}{4}p^2\kappa'. \quad . \quad . \quad (47)$$

The ratio of κ' to κ and the value of p are now determined by equations (37) and (47).

Without examining the expressions in general a numerical example will show the nature of the results.

Taking $n=10$, the appropriate roots of (37) and (47) are, approximately,

$$p=0{\cdot}8406\,; \quad \kappa'/\kappa=0{\cdot}897. \quad . \quad . \quad . \quad . \quad (48)$$

The following table shows the values of QQ′ and RR′ and of λ^2 for all the modes, calculated from (43) and (45); the values for $k=6, 7, 8, 9$ are omitted, as they are the same as for $k=4, 3, 2, 1$.

k.	QQ′.	RR′.	λ^2.	
0	0	−411	0	85
1	−61	−350	−318	−142
2	−177	−72	−307	−95
3	−292	−48	−545	−143
4	−369	−5	−475	−316
5	−396	0	−418	−418

We see that the motion is stable in all the possible modes with the exception of $k=0$. Reverting to (40), we find that $\kappa\lambda/4\pi a^2=2\pi\lambda/6{\cdot}3T$ approximately where T is the period of rotation of the rings in the steady state; thus the periods of the small oscillations in the stable modes range from about two-thirds to one-quarter of the period of rotation.

It is easily verified that $\lambda^2=0$ in the mode $k=0$ corresponds to a neutral displacement of the system, consisting of a rotation and dilatation of the rings without alteration of the ratio of their radii. On the other hand, the root $\lambda^2=85$ in this mode gives rise to definite instability.

9. The Karman vortex street may be obtained as a limiting case of the present problem. We make the radius a and the number of vortices n both become infinite, their ratio remaining finite. If the limiting value of $2\pi a/n$ is d, the distance between consecutive vortices in each row, and if h is the distance between the rows, we may put

$$p=(1+2\pi h/nd)^{-1} \; ; \; . \; . \; . \; . \; . \quad (49)$$

p approaches unity, while the limiting value of p^n is $e^{-2\pi h/d}$. We see from equations (37) and (47) that the ratio κ'/κ approaches unity, and (47) gives at once, in the limit, the Karman condition

$$\cosh^2(\pi h/d)=2. \; . \; . \; . \; . \; . \quad (50)$$

Further, if from (42) and (45) we write down the expressions for λ^2 when $k=0$, we find that for the positive root, $\kappa\lambda/4\pi a^2$ is of order $n^{-\frac{1}{2}}$; thus, as the limit is approached the instability in this mode merges with the neutral state in the same mode. It is only in this particular limiting case that we obtain a system which is completely stable.

LIII. *On the Motion of a Pendulum in a Viscous Medium.* *By* G. W. Brindley, *M.Sc., Assistant Lecturer in Physics, and* T. Emmerson, *University of Leeds* *.

(1) *Introduction.*

THE experiments described in this paper were undertaken in order to investigate the validity of Stokes's theory of the motion of a pendulum in a viscous medium, which has recently been questioned by Hoare †. Stokes ‡ considered the case of a sphere, suspended by an infinitely fine wire, performing small oscillations in a vertical plane, and assumed that the friction experienced by the sphere was directly proportional to its velocity. Stokes obtained an equation of motion for a sphere in an infinite medium, from which he deduced that the logarithmic decrement of the motion should be constant. Hoare found experimentally that the decrement was not constant for pendulums oscillating both in water and in aniline, and he drew the conclusion that Stokes's

* Communicated by Prof. R. Whiddington, F.R.S.

† F. E. Hoare, Phil. Mag. viii. p. 899 (1929); ix. p. 502 (1930).

‡ G. G. Stokes, Math. and Phys. Papers, iii p. 34.

assumption was not justifiable concerning the friction being proportional to the velocity.

It seemed worth while to investigate whether the results obtained by Hoare were really due to the incorrectness of Stokes's fundamental assumption, or whether they arose from the conditions required by the theory not being adequately satisfied. In our experiments particular attention has been given to the measurements of very small amplitudes for which Stokes's assumption is most likely to be satisfied, and the effect of the suspending wire and of the finite size of the medium have been investigated. Throughout these experiments the medium used was water.

The equation of motion of a pendulum consisting of a sphere of mass M and radius r, oscillating in a medium of density ρ and viscosity η, may be written

$$(\mathrm{M}+k\mathrm{M}')\frac{d^2x}{dt^2}=-(\mathrm{M}-\mathrm{M}')\frac{g}{l}x-\frac{9\pi}{2\mathrm{T}}k'\mathrm{M}'\frac{dx}{dt}, \quad . \quad (1)$$

where

$$k=\left(\frac{1}{2}+\frac{9}{4\beta r}\right),$$

$$k'=\left(\frac{1}{\beta r}+\frac{1}{\beta^2 r^2}\right),$$

and $$\beta=\sqrt{\frac{\pi\rho}{\eta\mathrm{T}}}.$$

Also

x = displacement of the sphere at time t,

T = periodic time in the medium,

M′ = mass of the medium displaced,

l = length of the pendulum,

g = acceleration due to gravity.

This equation neglects the finite arc of swing and assumes the resistance to motion due to the suspending wire to be negligible. The solution of equation (1) can be written

$$x=\mathrm{X}e^{-\frac{9\pi}{4\mathrm{T}}\frac{k'\mathrm{M}'}{(\mathrm{M}+k\mathrm{M}')}\cdot t}\cos(\theta-\gamma), \quad . \quad . \quad . \quad (2)$$

X and γ being constants depending on the initial conditions.

From equation (2) the logarithmic decrement, λ, is

$$\lambda=\frac{9\pi k'\mathrm{M}'}{8(\mathrm{M}+k\mathrm{M}')}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When we substitute for k and k' in equation (3) an *exact* simple expression for the coefficient of viscosity, η, cannot

be obtained, but if we neglect $\left(\frac{1}{\beta^2 r^2}\right)$, which is small compared with $\left(\frac{1}{\beta r}\right)$, and 2λ, which is small compared with π, then we obtain the following *approximate* expression for η, which is correct to 3 per cent.–4 per cent. for the conditions of our experiments :—

$$\eta = \frac{16r^2}{81\pi}\left[2\left(\frac{M}{M'}\right)+1\right]^2\left(\frac{\lambda^2}{T}\right). \quad . \quad . \quad . \quad . \quad (4)$$

An *exact* value for η can be calculated by substituting for k and k' in equation (3), and rearranging the terms, whence the following equation is obtained :—

$$\frac{4\lambda}{9\pi}\left(\frac{2M+M'}{M'}\right)\beta^2 r^2 - \left(1-\frac{2\lambda}{\pi}\right)\beta r - 1 = 0. \quad . \quad (5)$$

Having calculated βr, η can be obtained directly. The agreement between this experimental value of η and the generally accepted value gives a measure of the validity of Stokes's theory.

(2) *Experimental.*

A detailed account of the experimental arrangements is hardly necessary, for we have followed fairly closely the methods used by Hoare. The main experimental difficulty is that when a pendulum oscillates in water a large number of oscillations cannot easily be measured. Partly to test Stokes's theory over a wide range of conditions, and partly to obtain a larger number of oscillations, we have used heavier spheres than Hoare, our largest sphere having a diameter of $5\frac{3}{4}$ in.

To investigate the effect of the finite size of the tank containing the water four different tanks were used. The smallest was a cylindrical vessel, 11 in. in diameter and 15 in. deep. Two rectangular tanks were used, one being 18 in. by 12 in. and 15 in. deep, and the other 15 in. by 12 in. and 18 in. deep. A much larger tank of stout galvanized iron holding about 70 gallons of water was also used, this being approximately 3 feet long, 2 feet wide, and 2 feet deep.

The larger amplitudes from about 5 cm. down to 0·5 cm. were measured with a telescope, mounted on a solid iron base, having an eyepiece scale. Amplitudes smaller than 0·5 cm. could not be measured with sufficient accuracy with the telescope, and a low-power microscope, with an eyepiece scale, was therefore used. This enabled amplitudes as small as 0·02 cm. to be measured. To eliminate as far as possible

errors due to chance disturbances, the microscope was mounted rigidly on a massive marble slab. As the suspension wire appeared very broad in the microscope, the edge of the wire was observed ; by illuminating the field of view brightly, and keeping one side of the wire in the shadow, a sharp black edge was seen in the microscope, which, after a little practice, could be recorded accurately.

Care was taken throughout the experiments to keep the temperature of the water constant, the usual variation being not more than 0·5° C. from 15·5° C. There was a tendency during a series of observations for small air-bubbles to form on the sphere and on the suspension wire ; these were always removed. As the bulk of the water in the tank was disturbed slightly when the pendulum was set in motion, many oscillations were allowed to take place before readings were taken ; the surface of the water, as seen by reflexion, was always at rest when observations were made.

(3) *Results.*

(a) *Results for large Amplitudes* (5 cm.–0·5 cm.).

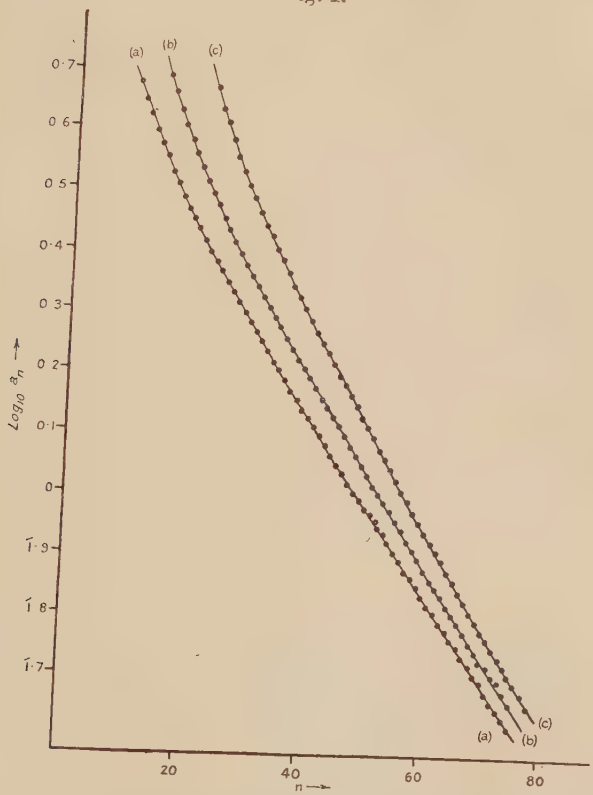
A series of experiments was made with a sphere of diameter 3 in., the amplitudes being measured with the telescope. The eyepiece scale was calibrated in cm., and for each length of pendulum the height of the point of observation on the wire above the centre of the sphere was determined, so that the actual amplitude of the centre of the sphere could be calculated. Curve (*a*) in fig. 1 is typical of many which were obtained, $\log_{10} a_n$ being plotted against n ; a is the amplitude of the centre of the sphere in cm. and n is the number of swings counted on one side of the zero only. From the slope of this curve the log-decrement is obtained directly.

It is seen that for amplitudes greater than 1 cm. the curve is certainly not linear, indicating that the log-decrement is not constant, a result which is in agreement with Hoare's observations. At smaller amplitudes between 1 cm. and 0·5 cm.—the smallest which could be measured with the telescope—the curve is practically linear, indicating a constant or almost constant log-decrement. The percentage accuracy of the measurements of these smallest amplitudes with the telescope was not great enough to determine with certainty whether the log-decrement is or is not constant, and this led to the microscope being used to measure small amplitudes more accurately.

(b) *Results for very small Amplitudes* (0·2 cm.–0·02 cm.).

When suitable precautions were taken to minimize chance disturbances amplitudes as small as 0·02 cm. could be measured. In all cases, the log-amplitude curve was found to be linear, indicating that at these amplitudes the log-decrement is constant. This is qualitatively in agreement

Fig. 1.



with Stokes's theory. A typical set of results is given in fig. 2.

To test equation (4) an extended series of measurements was made. For a particular sphere (λ^2/T) should be constant for pendulums of different lengths. The results obtained are given in Table I.

In Table I. the log-decrement λ is based on natural logarithms, and each value given is the mean of six or more determinations. (λ^2/T) is constant within the limits of

Fig. 2.

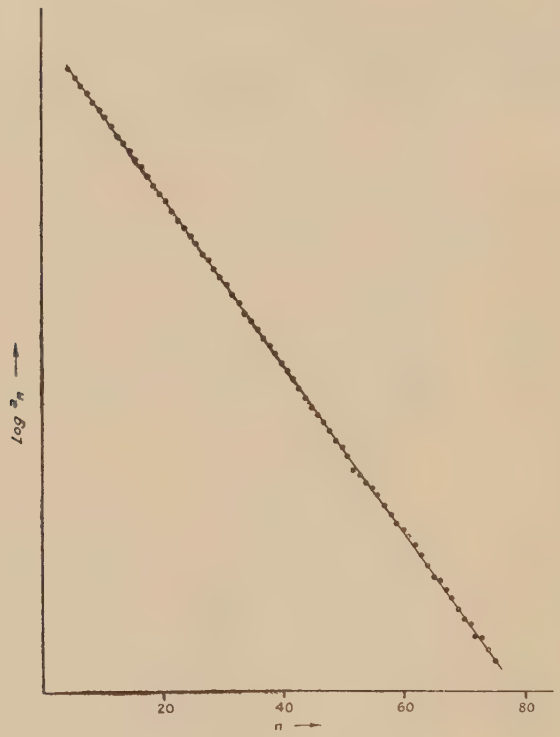


TABLE I.

Summary of results for 3 in. sphere, obtained with the microscope.

$M = 1808$ grams.
$M' = 231$,,
$r = 3{\cdot}81$ cm.

Period in seconds, T.	Length in cm.	λ.	$\left(\frac{\lambda^2}{T}\right)\times 1^{-5}$.
4·90	496	0·0159	5·15
4·64	432	0·0157	5·31
4·46	402	0·0152	5·22
4·13	344	0·0146	5·15
3·19	205	0·0128	5·12

experimental error, as equation (4) requires, but when η is calculated from the mean value of (λ^2/T) a value 0·0131 is obtained, which is much larger than the usually accepted value *, 0·0113.

There appeared to be two possible reasons for this high value for η : either the walls of the tank may restrict the flow of water round the oscillating sphere and thereby increase the damping, or the suspension wire may contribute appreciably to the damping of the pendulum. The effect of the finite size of the tank was investigated first.

(c) *Effect of the Finite Size of the Medium.*

The damping of the 3 in. sphere was determined for a particular length of pendulum in the four tanks described above, care being taken to keep the temperature of the water constant and to have the same height of water above the sphere in each tank. Some measurements were also made with the sphere oscillating in different directions. In every case the log-decrement was found to have the same value. This result might have been anticipated, because Stokes calculated the corrections which would have to be applied to equation (1) for a sphere of radius r oscillating at the centre of a spherical medium of radius b, and it follows from his results that the corrections become negligible when $(r/b)^3$ is small compared with unity, a condition satisfied in all our experiments. We cannot therefore attribute the high value obtained experimentally for η to the medium not being infinite.

(d) *Effect of the Suspension Wire.*

To obtain an estimate of the damping due to the wire passing through the water the following experiment was devised. Two additional wires, of the same material as the suspension wire, were soldered in small holes drilled in the sphere on opposite sides of the suspension wire and were mounted parallel to it by being soldered to the suspension wire at a point well above the surface of the water, the three wires lying in a plane normal to the direction of oscillation. The three wires were placed so far apart that the flow of water round each of them would not be disturbed by the neighbouring wires. The damping of the pendulum was then determined with one, two, and three wires ; curves (*a*), (*b*), and (*c*) in fig. 1 are typical results.

* Value for the viscosity of water at 15·5° C., International Critical Tables, v.

In Table II. are given the numerical results obtained with the microscope.

TABLE II.

Logarithmic decrement of a pendulum in water. Period = 4·91 sec., with one, two, and three suspension wires.

	λ.		No. of determinations.
Three wires	0·0173		13
		> ·0008	
Two wires	0·0165		9
		> ·0006	
One wire	0·0159		10
		> ·0007	
Zero wire (estimated) ...	[0·0152]		—

These results show that the damping due to the suspension wire is appreciable, and since each additional wire contributed approximately equally to the log-decrement, an estimate can be made of the decrement for an infinitely fine suspension wire.

Using equation (5) to obtain an exact value of η, and taking $\lambda = 0{\cdot}0152$ for an infinitely fine wire, we obtain

$$\eta = 0{\cdot}0114,$$

which is in excellent agreement with the generally accepted value, 0·0113.

We may therefore conclude that at amplitudes less than 0·2 *cm. the requirements of Stokes's theory are satisfied to within* 1 *per cent.*

TABLE III.

Summary of results for $5\frac{3}{4}$ in. sphere, obtained with the microscope.

M = 12,870 grams.
M′ = 1630 ,,
r = 7·30 cm.

Period of pendulum in water, T = 4·91 sec.

Log-decrement, with three wires,	$\lambda = 0{\cdot}0087_7$.
,, ,, two wires,	$\lambda = 0{\cdot}0084_5$.
,, ,, one wire,	$\lambda = 0{\cdot}0082_1$.
Estimated . . .. . ,, zero wire,	$\lambda = 0{\cdot}0078_7$.

(e) *A further Test of Stokes's Theory.*

In order to confirm the result obtained with the 3 in. sphere a larger sphere having a diameter of $5\frac{3}{4}$ in. was used, suspended by thicker wire having a diameter 0·071 cm. Since the damping of this sphere was much less than the damping of the 3 in. sphere, the log-decrement could be measured more accurately. The results obtained are given in Table III.

Then, using equation (5) and taking $\lambda = 0{\cdot}00787$ for an infinitely fine wire, we obtain

$$\eta = 0{\cdot}0116,$$

which is in close agreement with the value 0·0114 obtained with the 3 in. sphere.

(f) *Further Results.*

A series of results were also obtained with a sphere of diameter $2\frac{1}{2}$ in. suspended by fine wire of diameter 0·023 cm. These results are given below in Table IV.

TABLE IV.

Summary of results for $2\frac{1}{2}$ in. sphere, obtained with the microscope.

M = 1043 grams.
M′ = 133 „
r = 3·16 cm.
Period of pendulum in water, T = 4·96 sec.

Log-decrement, with three wires,		$\lambda = 0{\cdot}0225$	
			$> \cdot 0012$
„	„	two wires, $\lambda = 0{\cdot}0213$	
			$> \cdot 0013$
„	„	one wire, $\lambda = 0{\cdot}0200$	
			$> \cdot 0013$
Estimated „	„	zero wire, $\lambda = 0{\cdot}0187$	

Using equation (5) and taking $\lambda = 0{\cdot}0187$, we obtain

$$\eta = 0{\cdot}0116,$$

which is in agreement with the previous results.

(g) *Does the Variation of the Log-decrement with Amplitude depend on the Suspension Wire?*

An examination of the log-amplitude curves in fig. 1 shows that the curvature becomes more marked as the number of

suspension wires increases. This raises the question whether the log-amplitude curve would be linear up to larger amplitudes if a negligibly thin wire could be used. There does not appear to be any direct way of testing this, but the following result is not inconsistent with such a view. Apart from the disturbance created by the wire, the flow of water round the sphere is symmetrical about a horizontal plane through its centre. If, therefore, the sphere be half immersed, the flow of water round the lower half should not be altered, except possibly by some surface effects. The damping of a half-immersed sphere was determined experimentally, and the log-amplitude curve, though not quite linear, was found to be more nearly linear than the curve for the corresponding wholly-immersed sphere.

The evidence so far obtained shows that the variation of the log-decrement with amplitude depends to some extent on the suspension wire. This is in agreement with the result that the variation of the log-decrement is more marked for small spheres than for large spheres, for clearly the smaller the sphere the larger will be the relative effect of the wire.

(4) *Summary.*

The validity of Stokes's theory of the motion of a pendulum in a viscous medium is investigated. At amplitudes greater than 0·5 cm., for pendulums varying in length from 200 cm. to 500 cm., the log-decrement of the motion decreases as the amplitude becomes smaller, but at amplitudes smaller than 0·2 cm. the log-decrement is constant, as Stokes's theory requires. Moreover, when a correction is made for the damping due to the suspending wire, the value obtained from Stokes's theory for the coefficient of viscosity of water is in excellent agreement with the generally accepted value. Results are also given which suggest that the variation of the log-decrement at amplitudes greater than 0·5 cm. is connected with the suspension wire.

We therefore consider that Stokes's theory is correct and that his assumptions are justified, and that the modified theory proposed by Hoare (*loc. cit.*) is invalid.

Finally, we would like to express our thanks to Prof. Whiddington for his interest in these experiments, and to Dr. H. Jeffreys for reading through the manuscript.

The Physics Department,
The University of Leeds.
October 1930.

LIV. *Notices respecting New Books.*

Quantum Chemistry. A Short Introduction in Four Non-mathematical Lectures. By Arthur Haas, Ph.D. Translated by L. W. Codd, M.A. [Pp. ix+75.] (Constable & Co., Ltd., 1930. Price 6*s.* net.)

THE first of these lectures, "The Arithmetic of Chemical Periodicity," is a brief historical review of the development of spectral theory as regards quantum groups during the last twenty years. The exposition is lucid and contains no superfluous matter. In the second lecture, which deals with Valency interpreted by quantum theory, the impression is given that the facts of chemistry are clearly explained by London's theory of unpaired electrons. Actually only a very few simple cases can be adduced as applications of the theory, and only the fringe of the subject has as yet been touched upon. The outlook, however, seems definitely promising. The significance of homopolar combination from the standpoint of wave mechanics is treated with a better sense of proportion. The electronic grouping responsible for the Periodic Law is discussed in the third chapter, which includes an up-to-date table; and here again the setting out is admirable, with the exception of a small error of confusion on p. 45. The final chapter is by far the most interesting, partly because it touches on a great variety of topics, including nuclear quantum theory, and partly because it is somewhat speculative in character. The whole book is simple in expression, easy to read, and benefits from the obvious interest of the author both in the physical and in the chemical sides of the problems touched upon. The price is unfortunately high for such a small volume, and the printing of text on the highly glazed pages necessary for photographic reproduction does not add to its attractiveness. It is, however, a book to be recommended to all those who require a succinct presentation of the latest results in this field without the encumbrance of their derivation.

Handbuch der Experimentalphysik. Band 24, Erster Teil: *Allgemeine Physik der Röntgen-strahlen,* von F. Kirchner. (Akademische Verlagsgesellschaft, Leipzig. Geb. M. 55.)

Three volumes of this series of handbooks are devoted to the study of X-rays, crystal structure by Ott, spectroscopy of X-rays by Lindh, X-ray technique and the action of X-radiation on matter by Kirchner. The author has maintained the high standard of his predecessors and has presented a complete and authoritative treatise on this branch of

physics. The three sections of the book are clearly set out and illustrated, and describe very fully the different forms of X-ray tubes and their manipulation, the photoelectric effect and related phenomena, the Compton effect and the measurement of wave-lengths by interference and diffraction methods. This volume furnishes a clear account of recent theoretical and experimental work, including Dr. Kirchner's own researches and others of still more recent date, those of Coster, Larrson, and E. Jönsson. There are copious references to original papers, and, in spite of the bewildering progress made in this subject during the last few years, no research of importance has been unnoticed. Dr. Kirchner's book can be recommended as a valuable contribution to X-ray literature, although the high price of the volume will seriously limit the number of purchasers.

Introduction to Statistical Mechanics. By James Rice, M.A. (Constable & Co., Ltd., Orange Street, London, W.C.2. Price 18*s.* net.)

Prof. Rice's book gives a clear and able exposition of statistical mechanics and provides an admirable introduction to the study of more advanced works and original memoirs. Deferring the consideration of generalized dynamics, and employing only elementary dynamics in the first instance, the author smooths the way, arouses and maintains interest in the subject, and takes the reader by easy steps to more advanced problems. Among the topics thus treated may be mentioned the specific heats of gases and solids, the Nernst heat theorem, density fluctuations, collision frequency, and the kinetics of gas reactions. In the later chapters of the book the author introduces the quantum principles and the generalized dynamics of Lagrange, Hamilton, and Jacobi. In an appendix the statistical methods of Bose, Einstein, Fermi, and Dirac are expounded, with a brief account of the Darwin-Fowler method of partition functions. All students of physics and chemistry, especially private students, will be grateful to the author for providing a text-book which will be invaluable to them in the further study of statistical and quantum mechanics and in the appreciation of the advances made in recent research.

Applied Mathematics for Engineers. By T. Hodgson. (Chapman & Hall, 11 Henrietta Street, Covent Garden, London, W.C.2. Vol. I., 9*s.* 6*d.* net ; Vol. II., 13*s.* 6*d.* net.)

The first two volumes of Applied Mathematics are the outcome of the author's wide experience on the teaching staff of the City and Guilds Engineering College. The first volume,

where graphical and vectors methods are largely employed, is devoted to statics, mass centres, and the elementary theory of the planimeter, bending moments, force diagrams of rigid frames, cantilever, girder and roof trusses, with a passing reference to Pascal's Theorem, virtual work, and potential energy. The second volume, dynamics in two dimensions, introduces elementary differentiation and integration and the study of simple differential equations, which will receive fuller treatment in the third volume. These are applied to the rotation of rigid bodies about fixed axes, kinetic and potential energy problems, and motion due to impulses. A valuable feature of these books is the provision of six hundred exercises with answers at the end of the chapters, and many worked-out examples in the text.

Barlow's Tables. Third Edition. By L. J. Comrie, M.A., Ph.D. (E. and F. N. Spon, Ltd., 57 Haymarket, London, S.W.1. Price 7*s.* 6*d.* net.)

Addition and Subtraction Logarithms. By L. M. Berkeley. (White Book and Supply Co., 36 West 91st Street, New York City. Price $3.25.)

(1) Barlow's tables first appeared more than a century ago, and, in spite of the meagre calculating equipment at that time, was a noteworthy achievement, the author "being well aware that the utility of the performance depended wholly on its accuracy." Dr. Comrie has thoroughly revised and extended these tables and attained the highest degree of accuracy, the error in the last figure less than half a unit. The revised tables include squares, cubes : sq. roots to eight significant figures, $\sqrt{10n}$ to nine figures, cube roots to eight with first differences, reciprocals to seven figures from 1000 to 10,000. Other tables, omitted from the second edition, have been restored, fourth to tenth powers of the first hundred digits, eleventh to the twentieth powers of the first ten digits, and a number of constants involving e and π to fifteen places. The reasonable price of the third edition of "Barlow" should bring this indispensable set of tables within the reach of mathematicians, elementary and advanced, especially those engaged in the practice of computation.

(2) In 1802 Z. Leonelli communicated a paper to the Institut National des Sciences et Arts on the subject of addition and subtraction logarithms with short tables to fourteen places of decimals. Ten years later Gauss published his five-figure tables. Mr. Berkeley's five-figure tables are a great improvement on those constructed by earlier computers ; they have been arranged so that it is unnecessary to find interpolated

values, a step in the direction of simplification which will be welcomed by all computers. In the short introduction a description of the tables is given with several worked examples from geometry and spherical trigonometry, indicating the extreme simplicity and convenience of their use in extended calculations. Very extensive tables of addition and subtraction logarithms have recently appeared—tables fondamentales pour les logarithmes d'addition et de soustraction to sixteen places, with two subsidiary tables calculated by H. Andoyer, and published in the *Bulletin Astronomique*.

L'ancienne et la nouvelle théorie des quanta. Par Eugène Bloch. (Librairie Scientifique, Hermann et C^ie^, 6 Rue de la Sorbonne, Paris. Price 90 fr.)

Two books by Prof. Bloch, 'The Kinetic Theory of Gases' and 'Thermionic Phenomena,' are well known to readers, through their translation into English. The 'History and Development of the Quantum Theory' is another outstanding work from his pen, based largely upon lectures delivered in the Sorbonne. The topics selected for discussion, under the old theory, include atomic structure and the periodic table, the theory of Bohr and its application to the hydrogen spectrum, space quantization and the magneton, spectral lines and the spinning electron, and magneto-optics. The author then gives a concise account of the Hamiltonian dynamics, the elements of the wave theory of matter and its applications, matrix mechanics, and the modifications of the classical statistics due to Bose and Fermi. The experimental results of Davisson and Germer and others, a verification of the undulatory nature of the electron, are clearly described and illustrated. A bibliography of works dealing with particular applications of the quantum theory, X-rays and electrons, theories of magnetism, ionization, and resonance, etc., is provided for the benefit of readers whose interest has been stimulated to undertake more advanced study of the subject. Prof. Bloch's latest volume will prove a useful guide to students of modern physics, and can be heartily recommended to their notice.

Lord Balfour in his Relation to Science. By Lord Rayleigh. (Cambridge University Press, Fetter Lane, London. Price 2*s*. 6*d*. net.)

The obituary notice of Lord Balfour, who for more than forty years was a Fellow of the Royal Society, was written at the request of the Council of the Society by Lord Rayleigh. The publication of this notice in book form brings the account of Lord Balfour's administrative work on behalf of scientific, industrial, and medical research to the wider circle of readers

interested in the remarkable developments during the last three decades. Largely through his efforts the National Physical Laboratory was founded at the beginning of this period, the Department of Scientific and Industrial Research in the early years of the Great War, and later the reorganization of the scientific work of Government Departments, and the Chairmanship of the Research Committee of the Imperial Conference. By his enthusiasm and interest in the development of science he was one of the earliest advocates of its applications to industry, "science, the great instrument of social change, and in the promotion of the happiness and well-being of mankind." *Crescat scientia vita excolatur.*

Cours d'Optique. Par G. Bruhat, Professeur à la Faculté des Sciences de l'Université de Paris. [Pp. ix+756, with 657 figures.] (Paris: Masson & C^ie^, éditeurs, 1930. Price 100 fr.)

The aim of the author in writing this book has been to give a *résumé* of the essential experimental facts connected with optics and of the theoretical laws which connect them together. The orientation is thus essentially experimental rather than theoretical. Subjects whose interest is mainly of a theoretical nature, such as the effect of the motion of a body on optical phenomena, have been omitted. Geometrical optics, which is largely mathematical, has also been omitted to a large extent. Notwithstanding these omissions the volume is large. It covers a wide range of subjects in a very complete manner.

The first part is devoted to generalities, such as the propagation of light, reflexion and refraction, photometry, the velocity of light, vibrations and their composition, polarization, and interference. The second part deals with interference phenomena. This section is excellent and contains a good account of the various types of interferometers, their uses and applications. The third part deals with diffraction phenomena and includes the theory of the diffraction grating and a discussion of the resolving power of lenses, gratings, etc. The fourth part is concerned with the propagation of light in isotropic media; the electromagnetic theory of light is explained and is made the basis of the theoretical treatment of reflexion, refraction, polarization, and dispersion. Total reflexion and metallic reflexion are also discussed. Propagation in anisotropic media is considered in the fifth part. The theoretical treatment by the electromagnetic theory is given and applied to crystals, interference phenomena, rotation of the plane of polarization, etc. Applications, such as the quarter-wave plate, are included. The final part gives a comprehensive account of the more important results

of modern spectroscopic researches, bearing on X-ray spectra and optical spectra. Bohr's theory of the atom is explained and used as a basis, but an account of the reconciliation of the classical and modern theories by means of wave-mechanics is omitted as not being within the scope of the volume. This part includes a description of various types of spectroscopes and of spectra. The Zeeman and Stark effects are considered in some detail. The phenomena of fluorescence and phosphorescence are described and discussed.

The treatment is clear and logical; the volume is not overburdened with mathematics, but the mathematical sections are good. A large number of diagrams and figures are included. As is customary in French books, a detailed table of contents is given but there is no index, a regrettable omission. The printing is good.

Thermodynamic Relations in Multi-component Systems. By Roy W. Goranson. [Pp. xix+328.] (Carnegie Institution of Washington: Publication No. 408, 1930.)

This publication contains a tabulation of thermodynamic relations existing between the variable quantities of multi-component systems (including temperature, pressure, and the amounts of the substances present) in terms of quantities that can be readily obtained from experiment. It would not be possible to list the whole of the first and second derivatives of the variable quantities, but all which are considered to be of practical value have been evaluated. Tabulations of first and second derivatives for 162 groups are given, as follows:—All the component masses constant, 8 groups; all the component masses but one constant, 28 groups; all the component masses but two constant, 56 groups; all the component masses but three constant, 70 groups; The tables are thus a very considerable extension of any previous tables, and will be found of great value in experimental thermodynamical investigations.

The tables are preceded by an introduction which is essentially a mathematical outline of the structure of thermodynamics. This introduction is characterized by logical precision and mathematical exactitude. The fundamental ideas, undefined concepts, or directly negative catalysts (" anti-knocks ") are added in order to control the rate of combustion by controlling the length of the chain reactions, is not mentioned. The dependence of the quantum yield on temperature and wave-length is considered in some detail, and the final chapter is concerned with experimental methods.
